

Chapter 05.02

Direct Method of Interpolation

After reading this chapter, you should be able to:

1. *apply the direct method of interpolation,*
2. *solve problems using the direct method of interpolation, and*
3. *use the direct method interpolants to find derivatives and integrals of discrete functions.*

What is interpolation?

Many a times, data is given only at discrete points such as (x_0, y_0) , (x_1, y_1) , \dots , (x_{n-1}, y_{n-1}) , (x_n, y_n) . So, how then does one find the value of y at any other value of x ? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x . This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

- (A) evaluate,
- (B) differentiate, and
- (C) integrate

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. One of the methods of interpolation is called the direct method. Other methods include Newton's divided difference polynomial method and the Lagrangian interpolation method. We will discuss the direct method in this chapter.

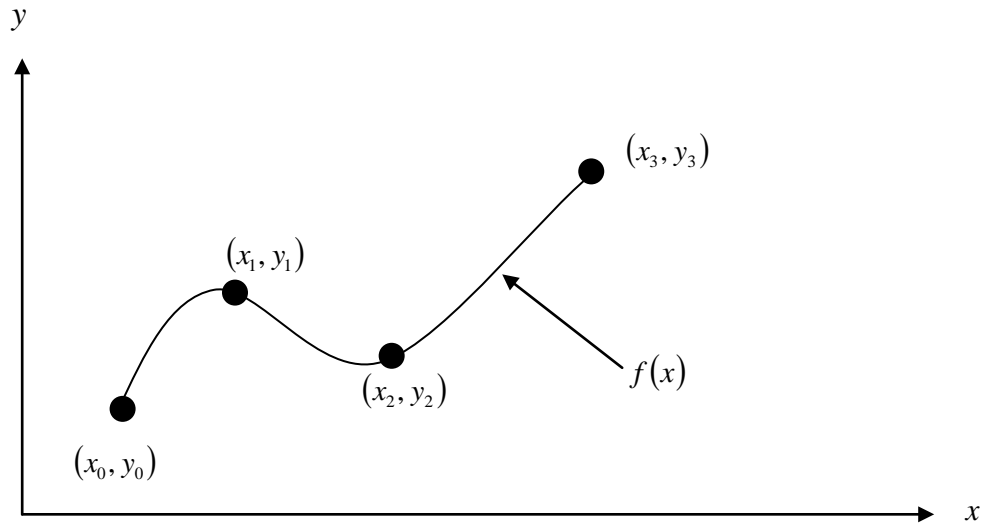


Figure 1 Interpolation of discrete data.

Direct Method

The direct method of interpolation is based on the following premise. Given $n + 1$ data points, fit a polynomial of order n as given below

$$y = a_0 + a_1x + \dots + a_nx^n \quad (1)$$

through the data, where a_0, a_1, \dots, a_n are $n + 1$ real constants. Since $n + 1$ values of y are given at $n + 1$ values of x , one can write $n + 1$ equations. Then the $n + 1$ constants, a_0, a_1, \dots, a_n can be found by solving the $n + 1$ simultaneous linear equations. To find the value of y at a given value of x , simply substitute the value of x in Equation 1.

But, it is not necessary to use all the data points. How does one then choose the order of the polynomial and what data points to use? This concept and the direct method of interpolation are best illustrated using examples.

Example 1

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15"×10" rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 1.

Table 1 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

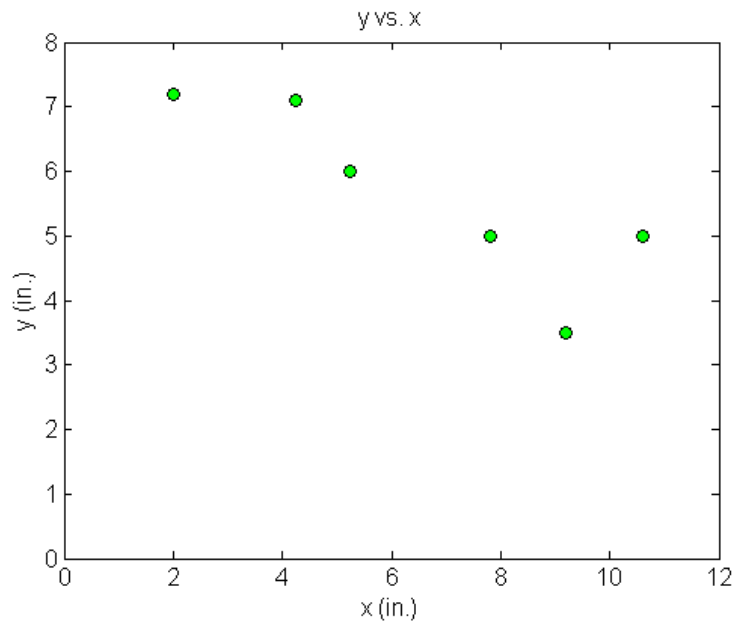


Figure 2 Location of holes on the rectangular plate.

If the laser is traversing from $x = 2.00$ to $x = 4.25$ in a linear path, what is the value of y at $x = 4.00$ using the direct method of interpolation and a first order polynomial?

Solution

For first order polynomial interpolation (also called linear interpolation), we choose the value of y given by

$$y(x) = a_0 + a_1x$$

y

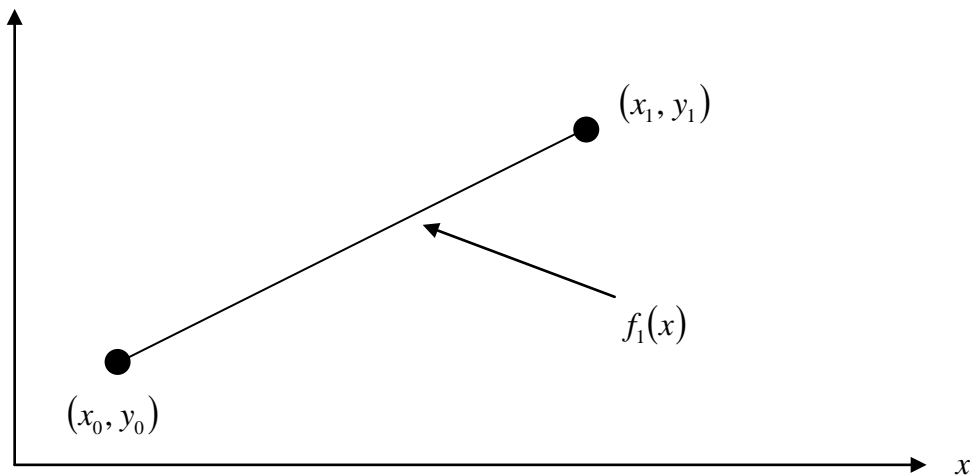


Figure 3 Linear interpolation.

Since we want to find the value of y at $x = 4.00$, using the two points $x_0 = 2.00$ and $x_1 = 4.25$, then

$$x_0 = 2.00, y(x_0) = 7.2$$

$$x_1 = 4.25, y(x_1) = 7.1$$

gives

$$y(2.00) = a_0 + a_1(2.00) = 7.2$$

$$y(4.25) = a_0 + a_1(4.25) = 7.1$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.00 \\ 1 & 4.25 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 7.2 \\ 7.1 \end{bmatrix}$$

Solving the above two equations gives

$$a_0 = 7.2889$$

$$a_1 = -0.044444$$

Hence

$$y(x) = a_0 + a_1x$$

$$y(x) = 7.2889 - 0.044444x, \quad 2.00 \leq x \leq 4.25$$

$$\begin{aligned} y(4.00) &= 7.2889 - 0.044444(4.00) \\ &= 7.1111 \text{ in.} \end{aligned}$$

Example 2

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15"×10" rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 2.

Table 2 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

If the laser is traversing from $x = 2.00$ to $x = 4.25$ to $x = 5.25$ in a quadratic path, what is the value of y at $x = 4.00$ using the direct method of interpolation and a second order polynomial? Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order polynomial interpolation (also called quadratic interpolation), we choose the value of y given by

$$y(x) = a_0 + a_1x + a_2x^2$$

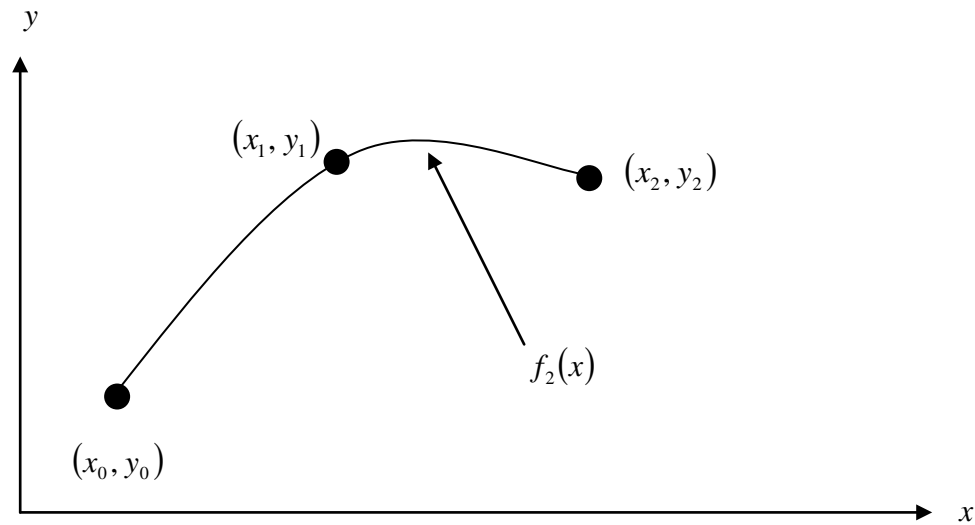


Figure 4 Quadratic interpolation.

Since we want to find the value of y at $x = 4.00$, using the three points as $x_0 = 2.00$, $x_1 = 4.25$ and $x_2 = 5.25$, then

$$x_0 = 2.00, \quad y(x_0) = 7.2$$

$$x_1 = 4.25, \quad y(x_1) = 7.1$$

$$x_2 = 5.25, \quad y(x_2) = 6.0$$

gives

$$y(2.00) = a_0 + a_1(2.00) + a_2(2.00)^2 = 7.2$$

$$y(4.25) = a_0 + a_1(4.25) + a_2(4.25)^2 = 7.1$$

$$y(5.25) = a_0 + a_1(5.25) + a_2(5.25)^2 = 6.0$$

Writing the three equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.00 & 4 \\ 1 & 4.25 & 18.063 \\ 1 & 5.25 & 27.563 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7.2 \\ 7.1 \\ 6.0 \end{bmatrix}$$

Solving the above three equations gives

$$a_0 = 4.5282$$

$$a_1 = 1.9855$$

$$a_2 = -0.32479$$

Hence

$$y(x) = 4.5282 + 1.9855x - 0.32479x^2, \quad 2.00 \leq x \leq 5.25$$

At $x = 4.00$,

$$\begin{aligned} y(4.00) &= 4.5282 + 1.9855(4.00) - 0.32479(4.00)^2 \\ &= 7.2735 \text{ in.} \end{aligned}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the first and second order polynomial is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{7.2735 - 7.1111}{7.2735} \right| \times 100 \\ &= 2.2327\% \end{aligned}$$

Example 3

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15"×10" rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 3.

Table 3 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

Find the path traversed through the six points using the direct method of interpolation and a fifth order polynomial.

Solution

For fifth order polynomial interpolation, also called quintic interpolation, we choose the value of y given by

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$$

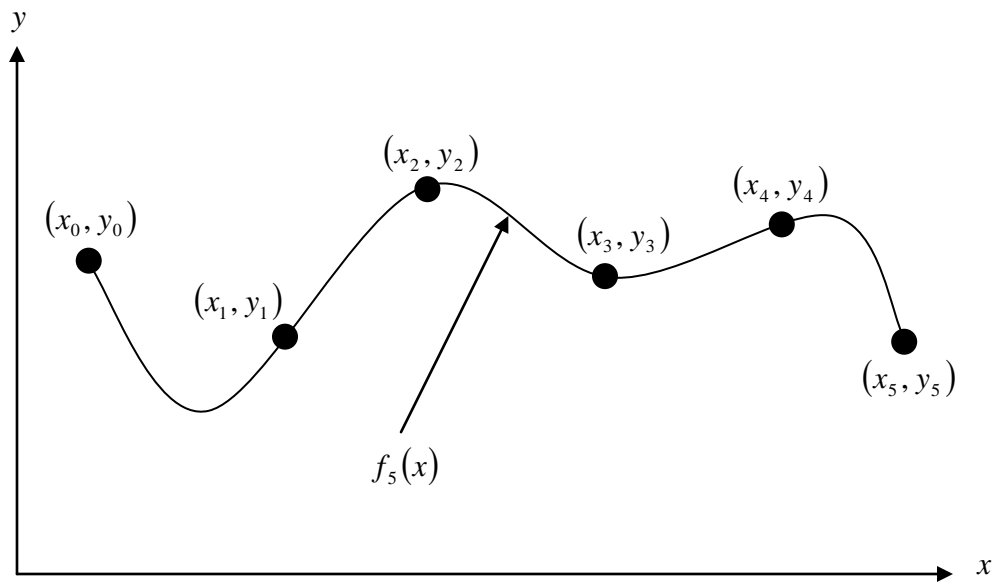


Figure 5 5th order polynomial interpolation.

Using the six points,

$$x_0 = 2.00, \quad y(x_0) = 7.2$$

$$x_1 = 4.25, \quad y(x_1) = 7.1$$

$$x_2 = 5.25, \quad y(x_2) = 6.0$$

$$x_3 = 7.81, \quad y(x_3) = 5.0$$

$$x_4 = 9.20, \quad y(x_4) = 3.5$$

$$x_5 = 10.60, \quad y(x_5) = 5.0$$

gives

$$y(2.00) = a_0 + a_1(2.00) + a_2(2.00)^2 + a_3(2.00)^3 + a_4(2.00)^4 + a_5(2.00)^5 = 7.2$$

$$y(4.25) = a_0 + a_1(4.25) + a_2(4.25)^2 + a_3(4.25)^3 + a_4(4.25)^4 + a_5(4.25)^5 = 7.1$$

$$y(5.25) = a_0 + a_1(5.25) + a_2(5.25)^2 + a_3(5.25)^3 + a_4(5.25)^4 + a_5(5.25)^5 = 6.0$$

$$y(7.81) = a_0 + a_1(7.81) + a_2(7.81)^2 + a_3(7.81)^3 + a_4(7.81)^4 + a_5(7.81)^5 = 5.0$$

$$y(9.20) = a_0 + a_1(9.20) + a_2(9.20)^2 + a_3(9.20)^3 + a_4(9.20)^4 + a_5(9.20)^5 = 3.5$$

$$y(10.60) = a_0 + a_1(10.60) + a_2(10.60)^2 + a_3(10.60)^3 + a_4(10.60)^4 + a_5(10.60)^5 = 5.0$$

Writing the six equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.00 & 4 & 8 & 16 & 32 \\ 1 & 4.25 & 18.063 & 76.766 & 326.25 & 1386.6 \\ 1 & 5.25 & 27.563 & 144.70 & 759.69 & 3988.4 \\ 1 & 7.81 & 60.996 & 476.38 & 3720.5 & 29057 \\ 1 & 9.20 & 84.64 & 778.69 & 7163.9 & 65908 \\ 1 & 10.60 & 112.36 & 1191.0 & 12625 & 133820 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 7.2 \\ 7.1 \\ 6.0 \\ 5.0 \\ 3.5 \\ 5.0 \end{bmatrix}$$

Solving the above six equations gives

$$a_0 = -30.898$$

$$a_1 = 41.344$$

$$a_2 = -15.855$$

$$a_3 = 2.7862$$

$$a_4 = -0.23091$$

$$a_5 = 0.0072923$$

Hence

$$\begin{aligned} y(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \\ &= -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 \\ &\quad - 0.23091x^4 + 0.0072923x^5, \quad 2 \leq x \leq 10.6 \end{aligned}$$

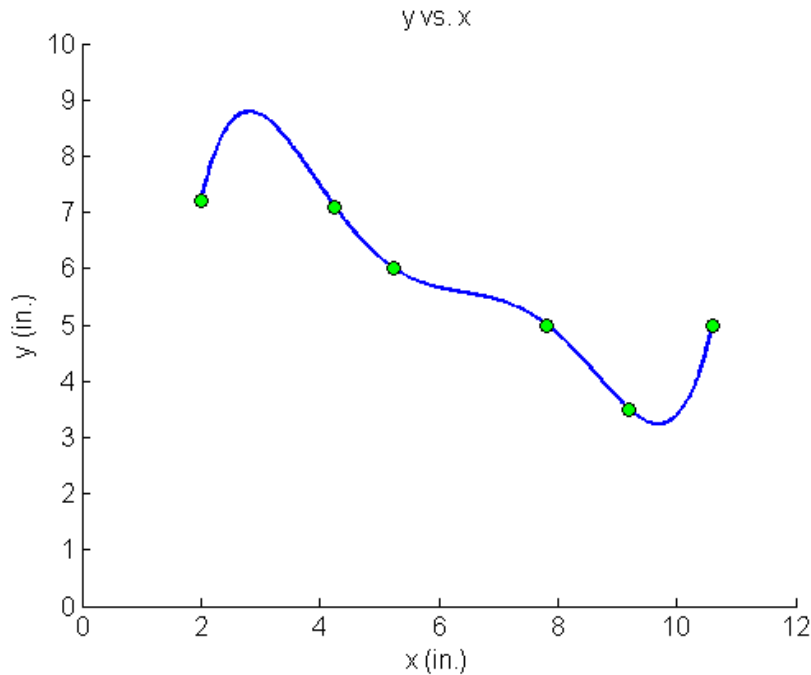


Figure 6 Fifth order polynomial to traverse points of robot path (using direct method of interpolation).

INTERPOLATION

Topic	Direct Method of Interpolation
Summary	Textbook notes on the direct method of interpolation.
Major	Computer Engineering
Authors	Autar Kaw, Peter Warr, Michael Keteltas
Date	August 27, 2009
Web Site	http://numericalmethods.eng.usf.edu
