

## Chapter 05.03

# Newton's Divided Difference Interpolation

*After reading this chapter, you should be able to:*

1. *derive Newton's divided difference method of interpolation,*
2. *apply Newton's divided difference method of interpolation, and*
3. *apply Newton's divided difference method interpolants to find derivatives and integrals.*

### What is interpolation?

Many a times, data is given only at discrete points such as  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $\dots$ ,  $(x_{n-1}, y_{n-1})$ ,  $(x_n, y_n)$ . So, how then does one find the value of  $y$  at any other value of  $x$ ? Well, a continuous function  $f(x)$  may be used to represent the  $n+1$  data values with  $f(x)$  passing through the  $n+1$  points (Figure 1). Then one can find the value of  $y$  at any other value of  $x$ . This is called interpolation.

Of course, if  $x$  falls outside the range of  $x$  for which the data is given, it is no longer interpolation but instead is called extrapolation.

So what kind of function  $f(x)$  should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

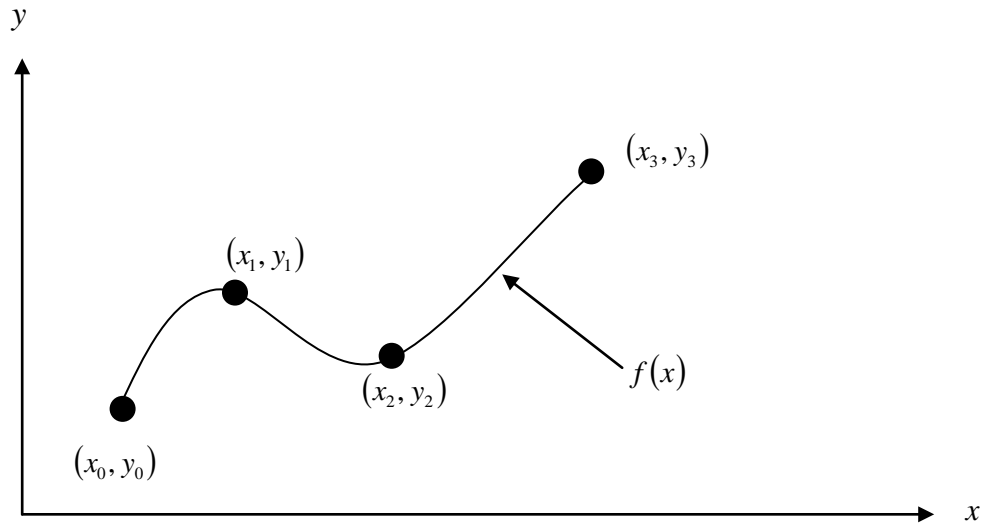
- (A) evaluate,
- (B) differentiate, and
- (C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order  $n$  that passes through the  $n+1$  points. One of the methods of interpolation is called Newton's divided difference polynomial method. Other methods include the direct method and the Lagrangian interpolation method. We discuss Newton's divided difference polynomial method in this chapter.

### Newton's Divided Difference Polynomial Method

To illustrate this method, linear and quadratic interpolation is presented first. Then, the general form of Newton's divided difference polynomial method is presented. To illustrate the general form, cubic interpolation is shown in Figure 1.



**Figure 1** Interpolation of discrete data.

### Linear Interpolation

Given  $(x_0, y_0)$  and  $(x_1, y_1)$ , fit a linear interpolant through the data. Noting  $y = f(x)$  and  $y_1 = f(x_1)$ , assume the linear interpolant  $f_1(x)$  is given by (Figure 2)

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at  $x = x_0$ ,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

and at  $x = x_1$ ,

$$\begin{aligned} f_1(x_1) &= f(x_1) = b_0 + b_1(x_1 - x_0) \\ &= f(x_0) + b_1(x_1 - x_0) \end{aligned}$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So

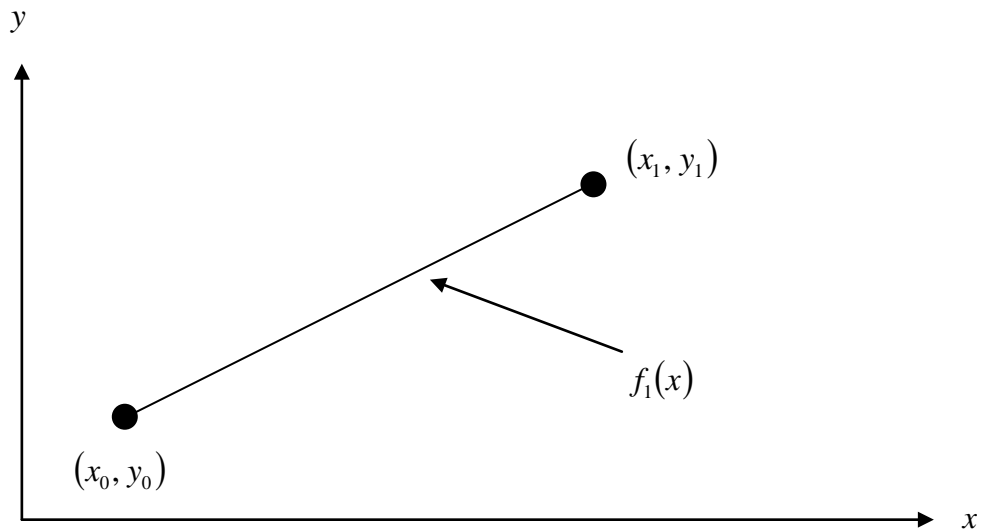
$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$



**Figure 2** Linear interpolation.

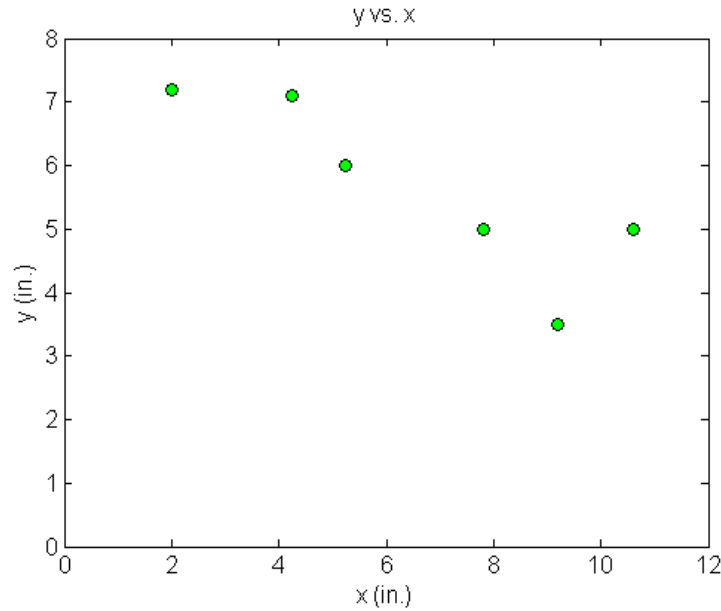
### Example 1

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15"×10" rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 1.

**Table 1** The coordinates of the holes on the plate.

$x$ (in.)	$y$ (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

If the laser is traversing from  $x = 2.00$  to  $x = 4.25$  in a linear path, what is the value of  $y$  at  $x = 4.00$  using Newton's divided difference method of interpolation and a first order polynomial?



**Figure 3** Location of holes on the rectangular plate.

### Solution

For linear interpolation, the value of  $y$  is given by

$$y(x) = b_0 + b_1(x - x_0)$$

Since we want to find the value of  $y$  at  $x = 4.00$ , using the two points  $x = 2.00$  and  $x = 4.25$ , then

$$x_0 = 2.00, \quad y(x_0) = 7.2$$

$$x_1 = 4.25, \quad y(x_1) = 7.1$$

gives

$$\begin{aligned} b_0 &= y(x_0) \\ &= 7.2 \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{y(x_1) - y(x_0)}{x_1 - x_0} \\ &= \frac{7.1 - 7.2}{4.25 - 2.00} \\ &= -0.044444 \end{aligned}$$

Hence

$$\begin{aligned} y(x) &= b_0 + b_1(x - x_0) \\ &= 7.2 - 0.044444(x - 2.00), \quad 2.00 \leq x \leq 4.25 \end{aligned}$$

At  $x = 4.00$

$$\begin{aligned} x(4.00) &= 7.2 - 0.044444(4.00 - 2.00) \\ &= 7.1111 \text{ in.} \end{aligned}$$

If we expand

$$y(x) = 7.2 - 0.0444444(x - 2.00), \quad 2.00 \leq x \leq 4.25$$

we get

$$y(x) = 7.2889 - 0.0444444x, \quad 2.00 \leq x \leq 4.25$$

This is the same expression that was obtained with the direct method.

### Quadratic Interpolation

Given  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$ , fit a quadratic interpolant through the data. Noting  $y = f(x)$ ,  $y_0 = f(x_0)$ ,  $y_1 = f(x_1)$ , and  $y_2 = f(x_2)$ , assume the quadratic interpolant  $f_2(x)$  is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At  $x = x_0$ ,

$$\begin{aligned} f_2(x_0) &= f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1) \\ &= b_0 \end{aligned}$$

$$b_0 = f(x_0)$$

At  $x = x_1$

$$\begin{aligned} f_2(x_1) &= f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1) \\ f(x_1) &= f(x_0) + b_1(x_1 - x_0) \end{aligned}$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At  $x = x_2$

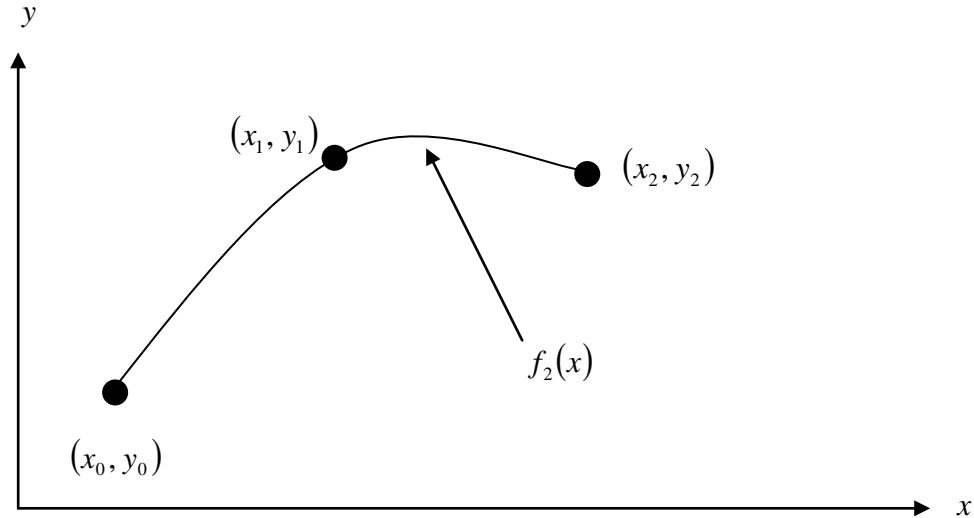
$$\begin{aligned} f_2(x_2) &= f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \\ f(x_2) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \end{aligned}$$

Giving

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Hence the quadratic interpolant is given by

$$\begin{aligned} f_2(x) &= b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \\ &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1) \end{aligned}$$



**Figure 4** Quadratic interpolation.

### Example 2

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15"×10" rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 2.

**Table 2** The coordinates of the holes on the plate.

$x$ (in.)	$y$ (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

If the laser is traversing from  $x = 2.00$  to  $x = 4.25$  to  $x = 5.25$  in a quadratic path, what is the value of  $y$  at  $x = 4.00$  using Newton's divided difference method of interpolation and a second order polynomial? Find the absolute relative approximate error for the second order polynomial approximation.

### Solution

For quadratic interpolation, the value of  $y$  is given by

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Since we want to find the value of  $y$  at  $x = 4.00$  and we are using a second order polynomial, we choose the three points as  $x_0 = 2.00$ ,  $x_1 = 4.25$ , and  $x_2 = 5.25$ .

Then

$$x_0 = 2.00, \quad y(x_0) = 7.2$$

$$x_1 = 4.25, \quad y(x_1) = 7.1$$

$$x_2 = 5.25, \quad y(x_2) = 6.0$$

gives

$$b_0 = y(x_0)$$

$$= 7.2$$

$$b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0}$$

$$= \frac{7.1 - 7.2}{4.25 - 2.00}$$

$$= -0.044444$$

$$b_2 = \frac{\frac{y(x_2) - y(x_1)}{x_2 - x_1} - \frac{y(x_1) - y(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{\frac{6.0 - 7.1}{5.25 - 4.25} - \frac{7.1 - 7.2}{4.25 - 2.00}}{5.25 - 2.00}$$

$$= \frac{-1.1 + 0.044444}{3.25}$$

$$= -0.32479$$

then

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= 7.2 - 0.044444(x - 2.00) - 0.32479(x - 2.00)(x - 4.25), \quad 2.00 \leq x \leq 5.25$$

At  $x = 4.00$

$$y(4.00) = 7.2 - 0.044444(4.00 - 2.00) - 0.32479(4.00 - 2.00)(4.00 - 4.25) \\ = 7.2735 \text{ in.}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the first and second order polynomial is

$$|\epsilon_a| = \left| \frac{7.2735 - 7.1111}{7.2735} \right| \times 100 \\ = 2.2327\%$$

If we expand,

$$y(x) = 7.2 - 0.044444(x - 2.00) - 0.32479(x - 2.00)(x - 4.25), \quad 2.00 \leq x \leq 5.25$$

we get

$$y(x) = 4.5282 + 1.9855x - 0.32479x^2, \quad 2.00 \leq x \leq 5.25$$

This is the same expression that was obtained with the direct method.

### General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for Newton's divided difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Note that  $b_0$ ,  $b_1$ , and  $b_2$  are finite divided differences.  $b_0$ ,  $b_1$ , and  $b_2$  are the first, second, and third finite divided differences, respectively. We denote the first divided difference by

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

where  $f[x_0]$ ,  $f[x_1, x_0]$ , and  $f[x_2, x_1, x_0]$  are called bracketed functions of their variables enclosed in square brackets.

Rewriting,

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for  $n + 1$  data points,  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , as

$$f_n(x) = b_0 + b_1(x - x_0) + \dots + b_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

$$b_{n-1} = f[x_{n-1}, x_{n-2}, \dots, x_0]$$

$$b_n = f[x_n, x_{n-1}, \dots, x_0]$$

where the definition of the  $m^{\text{th}}$  divided difference is

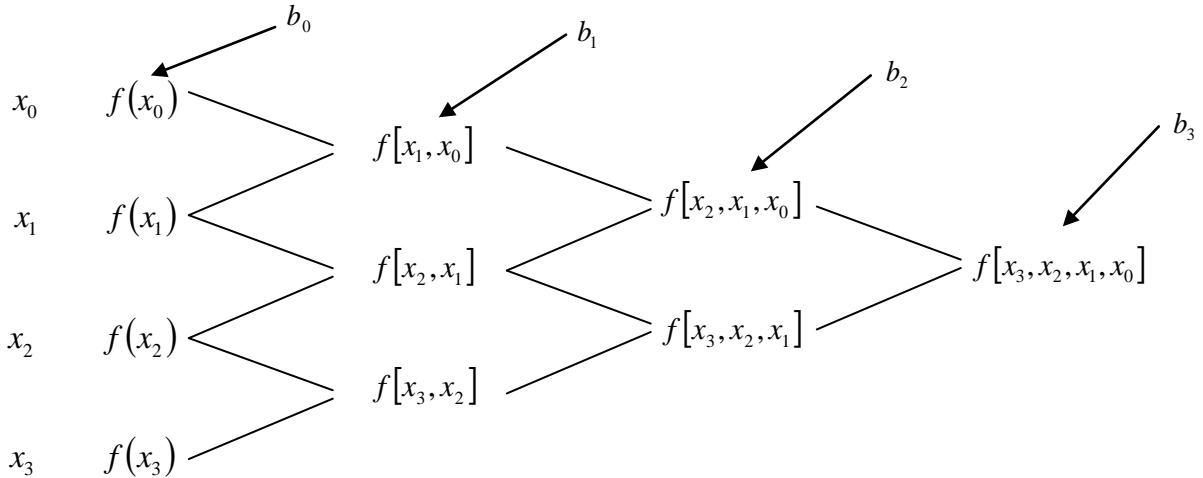
$$b_m = f[x_m, \dots, x_0]$$

$$= \frac{f[x_m, \dots, x_1] - f[x_{m-1}, \dots, x_0]}{x_m - x_0}$$

From the above definition, it can be seen that the divided differences are calculated recursively.

For an example of a third order polynomial, given  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ ,

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



**Figure 5** Table of divided differences for a cubic polynomial.

**Example 3**

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15"×10" rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 3.

**Table 3** The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

Find the path traversed through the six points using Newton's divided difference method of interpolation and a fifth order polynomial.

**Solution**

For a fifth order polynomial, the value of y is given by

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) + b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

Using the six points,

$$x_0 = 2.00, \quad y(x_0) = 7.2$$

$$x_1 = 4.25, \quad y(x_1) = 7.1$$

$$x_2 = 5.25, \quad y(x_2) = 6.0$$

$$x_3 = 7.81, \quad y(x_3) = 5.0$$

$$x_4 = 9.20, \quad y(x_4) = 3.5$$

$$x_5 = 10.60, \quad y(x_5) = 5.0$$

gives

$$b_0 = y[x_0]$$

$$= y(x_0)$$

$$= 7.2$$

$$b_1 = y[x_1, x_0]$$

$$= \frac{y(x_1) - y(x_0)}{x_1 - x_0}$$

$$= \frac{7.1 - 7.2}{4.25 - 2.00}$$

$$= -0.0444444$$

$$b_2 = y[x_2, x_1, x_0]$$

$$= \frac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0}$$

$$y[x_2, x_1] = \frac{y(x_2) - y(x_1)}{x_2 - x_1}$$

$$= \frac{6.0 - 7.1}{5.25 - 4.25}$$

$$= -1.1$$

$$y[x_1, x_0] = -0.0444444$$

$$b_2 = \frac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{-1.1 + 0.0444444}{5.25 - 2.00}$$

$$= -0.32479$$

$$b_3 = y[x_3, x_2, x_1, x_0]$$

$$= \frac{y[x_3, x_2, x_1] - y[x_2, x_1, x_0]}{x_3 - x_0}$$

$$y[x_3, x_2, x_1] = \frac{y[x_3, x_2] - y[x_2, x_1]}{x_3 - x_1}$$

$$\begin{aligned} y[x_3, x_2] &= \frac{y(x_3) - y(x_2)}{x_3 - x_2} \\ &= \frac{5.0 - 6.0}{7.81 - 5.25} \\ &= -0.39063 \end{aligned}$$

$$\begin{aligned} y[x_2, x_1] &= \frac{y(x_2) - y(x_1)}{x_2 - x_1} \\ &= \frac{6.0 - 7.1}{5.25 - 4.25} \\ &= -1.1 \end{aligned}$$

$$\begin{aligned} y[x_3, x_2, x_1] &= \frac{y[x_3, x_2] - y[x_2, x_1]}{x_3 - x_1} \\ &= \frac{-0.39063 + 1.1}{7.81 - 4.25} \\ &= 0.19926 \end{aligned}$$

$$y[x_2, x_1, x_0] = -0.32479$$

$$\begin{aligned} b_3 &= y[x_3, x_2, x_1, x_0] \\ &= \frac{y[x_3, x_2, x_1] - y[x_2, x_1, x_0]}{x_3 - x_0} \\ &= \frac{0.19926 + 0.32479}{7.81 - 2.00} \\ &= 0.090198 \end{aligned}$$

$$\begin{aligned} b_4 &= y[x_4, x_3, x_2, x_1, x_0] \\ &= \frac{y[x_4, x_3, x_2, x_1] - y[x_3, x_2, x_1, x_0]}{x_4 - x_0} \end{aligned}$$

$$y[x_4, x_3, x_2, x_1] = \frac{y[x_4, x_3, x_2] - y[x_3, x_2, x_1]}{x_4 - x_1}$$

$$y[x_4, x_3, x_2] = \frac{y[x_4, x_3] - y[x_3, x_2]}{x_4 - x_2}$$

$$\begin{aligned} y[x_4, x_3] &= \frac{y(x_4) - y(x_3)}{x_4 - x_3} \\ &= \frac{3.5 - 5.0}{9.20 - 7.81} \\ &= -1.0791 \end{aligned}$$

$$y[x_3, x_2] = -0.39063$$

$$y[x_4, x_3, x_2] = \frac{y[x_4, x_3] - y[x_3, x_2]}{x_4 - x_2}$$

$$= \frac{-1.0791 + 0.39063}{9.20 - 5.25}$$

$$= -0.17431$$

$$y[x_3, x_2, x_1] = 0.19926$$

$$y[x_4, x_3, x_2, x_1] = \frac{y[x_4, x_3, x_2] - y[x_3, x_2, x_1]}{x_4 - x_1}$$

$$= \frac{-0.17431 - 0.19926}{9.20 - 4.25}$$

$$= -0.075469$$

$$y[x_3, x_2, x_1, x_0] = 0.090198$$

$$b_4 = \frac{y[x_4, x_3, x_2, x_1] - y[x_3, x_2, x_1, x_0]}{x_4 - x_0}$$

$$= \frac{-0.075469 - 0.090198}{9.20 - 2.00}$$

$$= -0.023009$$

$$b_5 = y[x_5, x_4, x_3, x_2, x_1, x_0]$$

$$= \frac{y[x_5, x_4, x_3, x_2, x_1] - y[x_4, x_3, x_2, x_1, x_0]}{x_5 - x_0}$$

$$y[x_5, x_4, x_3, x_2, x_1] = \frac{y[x_5, x_4, x_3, x_2] - y[x_4, x_3, x_2, x_1]}{x_5 - x_1}$$

$$y[x_5, x_4, x_3, x_2] = \frac{y[x_5, x_4, x_3] - y[x_4, x_3, x_2]}{x_5 - x_2}$$

$$y[x_5, x_4, x_3] = \frac{y[x_5, x_4] - y[x_4, x_3]}{x_5 - x_3}$$

$$y[x_5, x_4] = \frac{y(x_5) - y(x_4)}{x_5 - x_4}$$

$$= \frac{5.0 - 3.5}{10.60 - 9.20}$$

$$= 1.0714$$

$$y[x_4, x_3] = -1.0791$$

$$y[x_5, x_4, x_3] = \frac{y[x_5, x_4] - y[x_4, x_3]}{x_5 - x_3}$$

$$= \frac{1.0714 + 1.0791}{10.60 - 7.81}$$

$$= 0.77081$$

$$y[x_4, x_3, x_2] = -0.17431$$

$$\begin{aligned} y[x_5, x_4, x_3, x_2] &= \frac{y[x_5, x_4, x_3] - y[x_4, x_3, x_2]}{x_5 - x_2} \\ &= \frac{0.77081 + 0.17431}{10.60 - 5.25} \\ &= 0.17666 \end{aligned}$$

$$y[x_4, x_3, x_2, x_1] = -0.075469$$

$$\begin{aligned} y[x_5, x_4, x_3, x_2, x_1] &= \frac{y[x_5, x_4, x_3, x_2] - y[x_4, x_3, x_2, x_1]}{x_5 - x_1} \\ &= \frac{0.17666 + 0.075469}{10.60 - 4.25} \\ &= 0.039705 \end{aligned}$$

$$y[x_4, x_3, x_2, x_1, x_0] = -0.023009$$

$$\begin{aligned} b_5 &= \frac{y[x_5, x_4, x_3, x_2, x_1] - y[x_4, x_3, x_2, x_1, x_0]}{x_5 - x_0} \\ &= \frac{0.039705 + 0.023009}{10.60 - 2.00} \\ &= 0.0072923 \end{aligned}$$

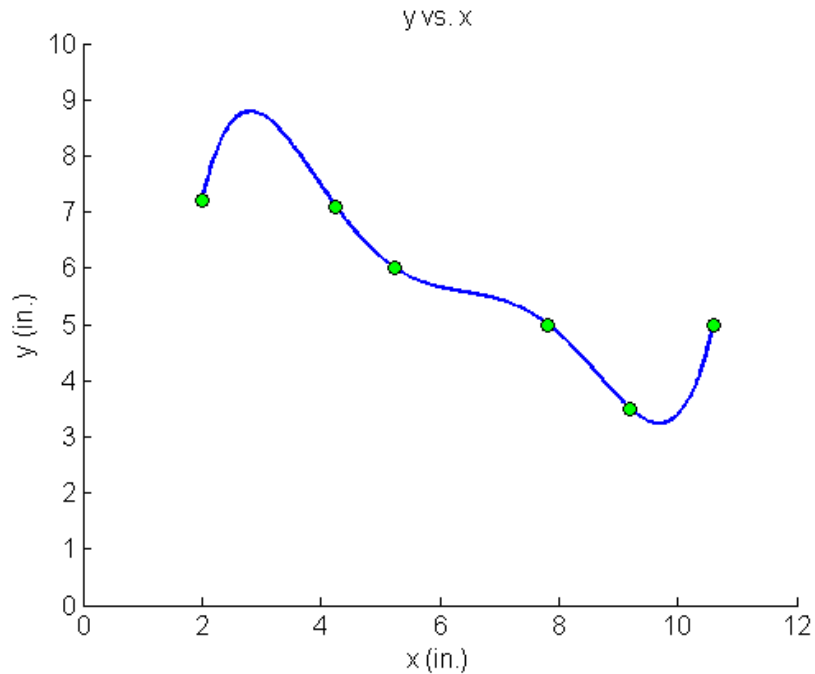
Hence

$$\begin{aligned} y(x) &= b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) \\ &\quad + b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) \\ &\quad + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4) \\ &= 7.2 - 0.04444(x - 2) - 0.32479(x - 2)(x - 4.25) \\ &\quad + 0.090198(x - 2)(x - 4.25)(x - 5.25) \\ &\quad - 0.023009(x - 2)(x - 4.25)(x - 5.25)(x - 7.81) \\ &\quad + 0.0072923(x - 2)(x - 4.25)(x - 5.25)(x - 7.81)(x - 9.2) \end{aligned}$$

Expanding this formula, we get

$$\begin{aligned} y(x) &= -30.898 + 41.344x - 15.855x^2 \\ &\quad + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, \quad 2 \leq x \leq 10.6 \end{aligned}$$

This is the same expression that was obtained with the direct method.



**Figure 6** Fifth order polynomial to traverse points of robot path (using direct method of interpolation).

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#### INTERPOLATION

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Topic	Newton's Divided Difference Interpolation
Summary	Textbook notes on Newton's divided difference interpolation.
Major	Computer Engineering
Authors	Autar Kaw, Michael Keteltas
Last Revised	August 27, 2009
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>

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