

## Chapter 05.05

# Spline Method of Interpolation

After reading this chapter, you should be able to:

1. interpolate data using spline interpolation, and
2. understand why spline interpolation is important.

### What is interpolation?

Many times, data is given only at discrete points such as  $(x_0, y_0)$ ,  $(x_1, y_1)$ , .....,  $(x_{n-1}, y_{n-1})$ ,  $(x_n, y_n)$ . So, how then does one find the value of  $y$  at any other value of  $x$ ? Well, a continuous function  $f(x)$  may be used to represent the  $n + 1$  data values with  $f(x)$  passing through the  $n + 1$  points (Figure 1). Then one can find the value of  $y$  at any other value of  $x$ . This is called *interpolation*.

Of course, if  $x$  falls outside the range of  $x$  for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function  $f(x)$  should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

- (A) evaluate,
- (B) differentiate, and
- (C) integrate

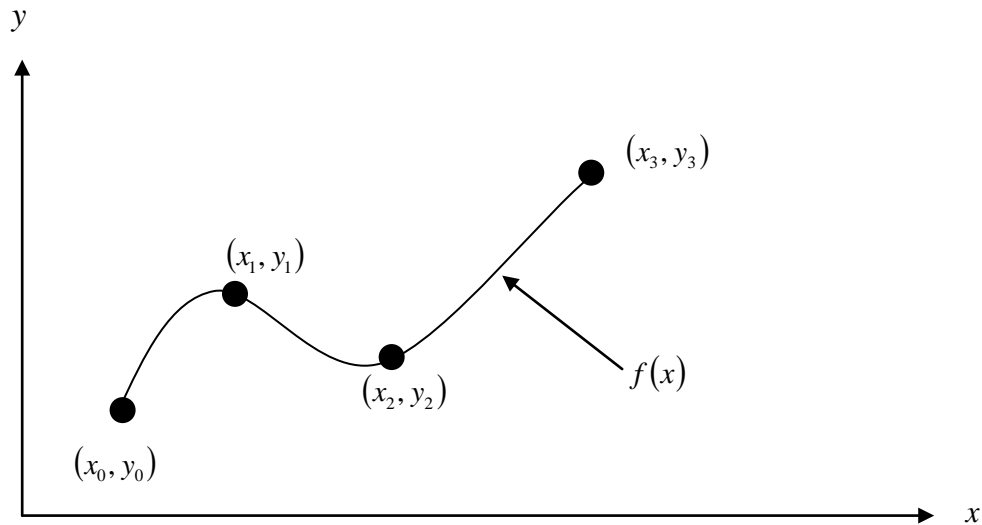
relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order  $n$  that passes through the  $n + 1$  points. Several methods to obtain such a polynomial include the direct method, Newton's divided difference polynomial method and the Lagrangian interpolation method.

So is the spline method yet another method of obtaining this  $n^{\text{th}}$  order polynomial. .... NO! Actually, when  $n$  becomes large, in many cases, one may get oscillatory behavior in the resulting polynomial. This was shown by Runge when he interpolated data based on a simple function of

$$y = \frac{1}{1 + 25x^2}$$

on an interval of  $[-1, 1]$ . For example, take six equidistantly spaced points in  $[-1, 1]$  and find  $y$  at these points as given in Table 1.



**Figure 1** Interpolation of discrete data.

**Table 1** Six equidistantly spaced points in  $[-1, 1]$ .

$x$	$y = \frac{1}{1 + 25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

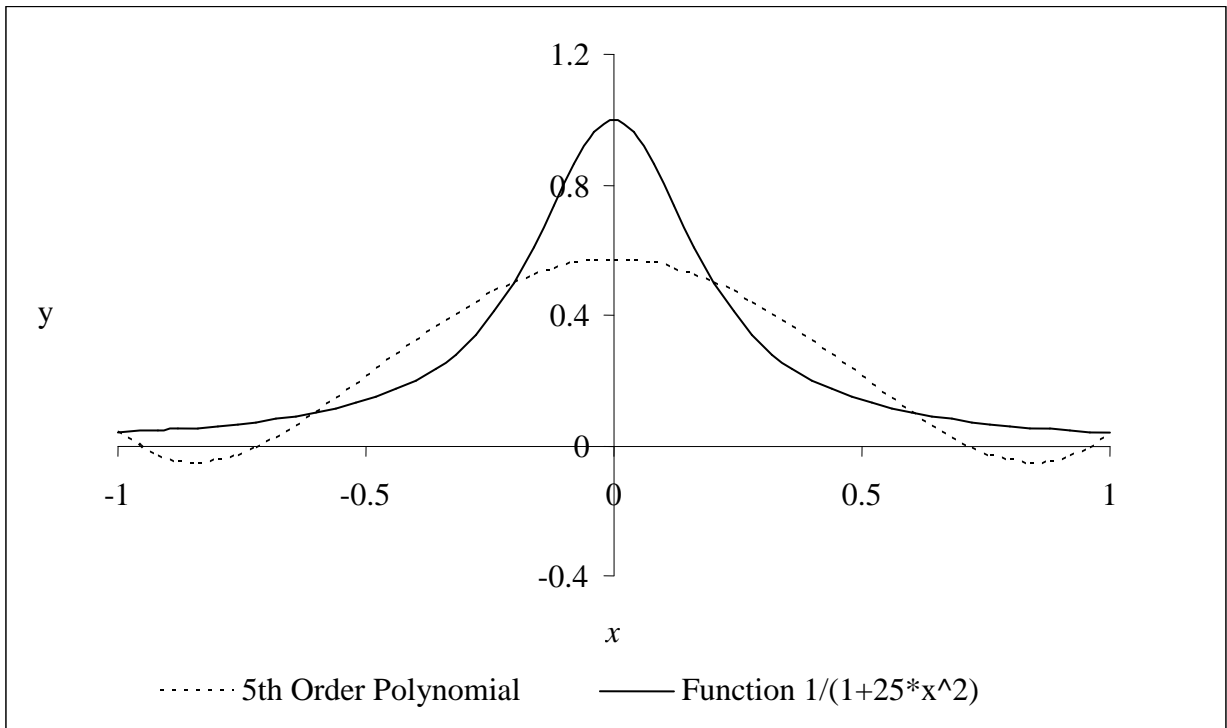
Now through these six points, one can pass a fifth order polynomial

$$f_5(x) = 3.1378 \times 10^{-11} x^5 + 1.2019 x^4 - 3.3651 \times 10^{-11} x^3 - 1.7308 x^2 + 1.0004 \times 10^{-11} x + 5.6731 \times 10^{-1},$$

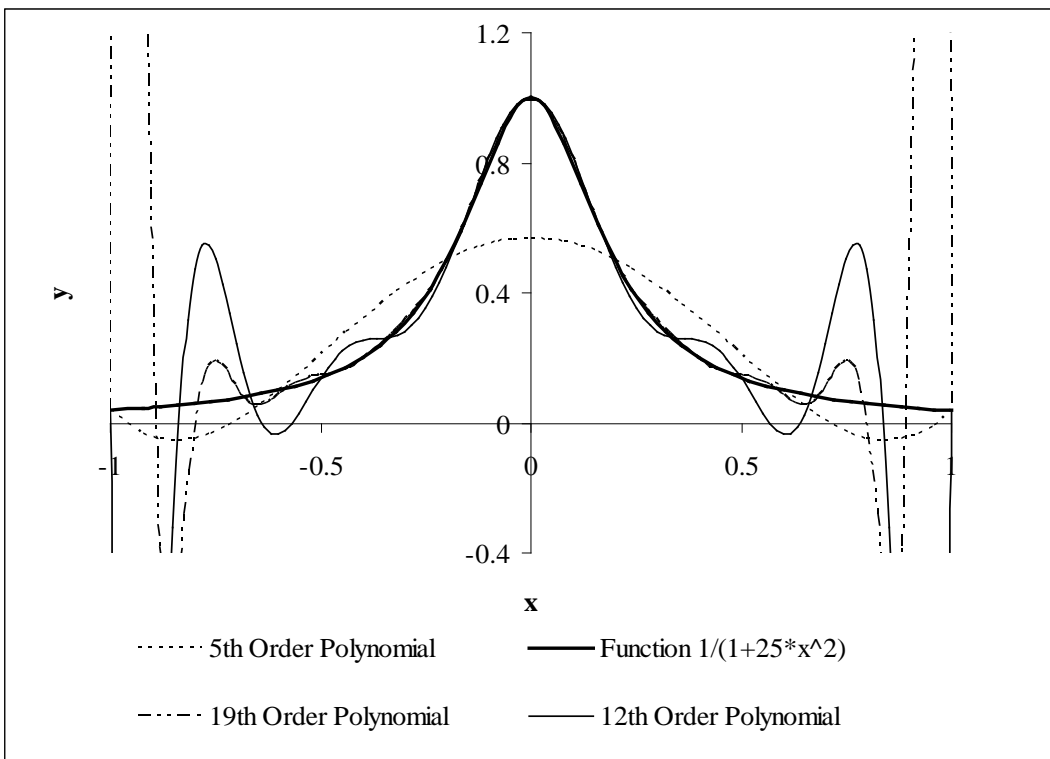
$$-1 \leq x \leq 1$$

through the six data points. On plotting the fifth order polynomial (Figure 2) and the original function, one can see that the two do not match well. One may consider choosing more points in the interval  $[-1, 1]$  to get a better match, but it diverges even more (see Figure 3), where 20 equidistant points were chosen in the interval  $[-1, 1]$  to draw a 19th order polynomial. In fact, Runge found that as the order of the polynomial becomes infinite, the polynomial diverges in the interval of  $-1 < x < -0.726$  and  $0.726 < x < 1$ .

So what is the answer to using information from more data points, but at the same time keeping the function true to the data behavior? The answer is in spline interpolation. The most common spline interpolations used are linear, quadratic, and cubic splines.



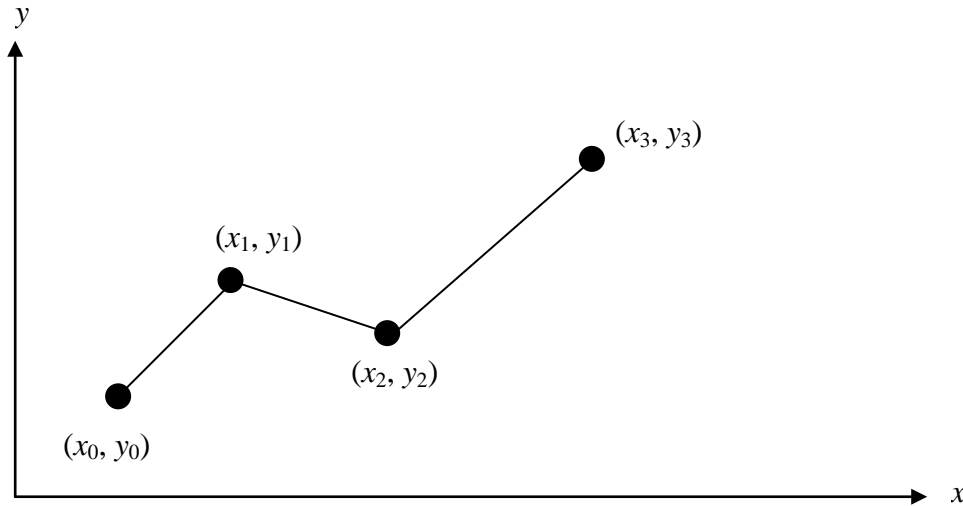
**Figure 2** 5th order polynomial interpolation with six equidistant points.



**Figure 3** Higher order polynomial interpolation is a bad idea.

### Linear Spline Interpolation

Given  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , fit linear splines (Figure 4) to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by  $y_i = f(x_i)$ .



**Figure 4** Linear splines.

$$\begin{aligned}
 f(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), & x_0 \leq x \leq x_1 \\
 &= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), & x_1 \leq x \leq x_2 \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 &= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), & x_{n-1} \leq x \leq x_n
 \end{aligned}$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

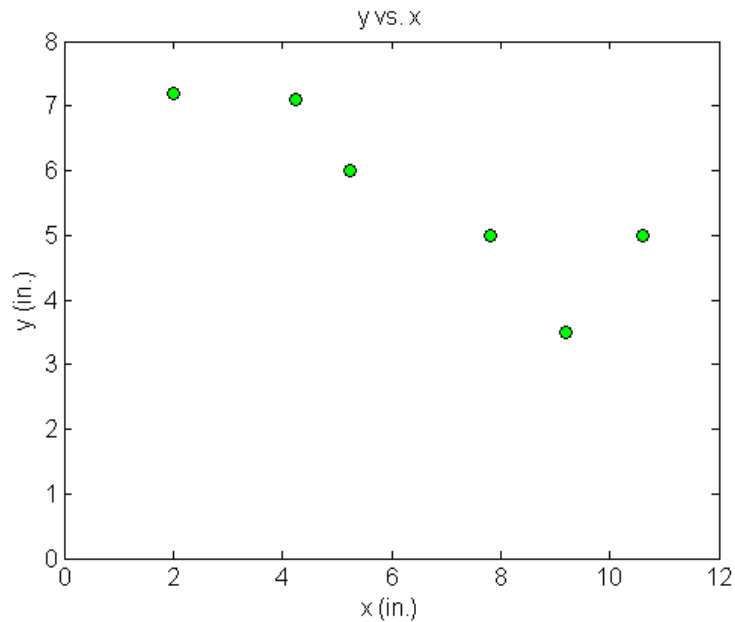
in the above function are simply slopes between  $x_{i-1}$  and  $x_i$ .

### Example 1

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15"×10" rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 2.

**Table 2** The coordinates of the holes on the plate.

$x$ (in.)	$y$ (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

**Figure 5** Location of the holes on the rectangle plate.

- If the laser is traversing from  $x = 2.00$  to  $x = 4.25$  in a linear path, what is the value of  $y$  at  $x = 4.00$  using linear splines?
- Find the path of the robot if it follows linear splines.
- Find the length of the path traversed by the robot following linear splines.

**Solution**

a) Since we want to find the value of  $y$  at  $x = 4.00$ , we need to choose the two data points closest to  $x = 4.00$  that also bracket  $x = 4.00$  to evaluate it. The two points are  $x_0 = 2.00$  and  $x_1 = 4.25$ .

Then

$$x_0 = 2.00, \quad y(x_0) = 7.2$$

$$x_1 = 4.25, \quad y(x_1) = 7.1$$

gives

$$y(x) = y(x_0) + \frac{y(x_1) - y(x_0)}{x_1 - x_0}(x - x_0)$$

$$= 7.2 + \frac{7.1 - 7.2}{4.25 - 2.00}(x - 2.00)$$

Hence

$$y(x) = 7.2 - 0.0444444(x - 2.00), \quad 2.00 \leq x \leq 4.25$$

At  $x = 4$ ,

$$\begin{aligned} y(4.00) &= 7.2 - 0.0444444(4.00 - 2.00) \\ &= 7.1111 \text{ in.} \end{aligned}$$

b) We have already obtained the linear spline connecting  $x = 2.00$  and  $x = 4.25$ .

$$y(x) = 7.2 - 0.0444444(x - 2.00), \quad 2.00 \leq x \leq 4.25 \quad (1)$$

The rest of the splines are as follows:

$$\begin{aligned} y(x) &= 7.1 + \frac{6.0 - 7.1}{5.25 - 4.25}(x - 4.25) \\ &= 7.1 - 1.1(x - 4.25), \quad 4.25 \leq x \leq 5.25 \end{aligned} \quad (2)$$

$$\begin{aligned} y(x) &= 6.0 + \frac{5.0 - 6.0}{7.81 - 5.25}(x - 5.25) \\ &= 6.0 - 0.39063(x - 5.25), \quad 5.25 \leq x \leq 7.81 \end{aligned} \quad (3)$$

$$\begin{aligned} y(x) &= 5.0 + \frac{3.5 - 5.0}{9.20 - 7.81}(x - 7.81) \\ &= 5.0 - 1.0791(x - 7.81), \quad 7.81 \leq x \leq 9.20 \end{aligned} \quad (4)$$

$$\begin{aligned} y(x) &= 3.5 + \frac{5.0 - 3.5}{10.60 - 9.20}(x - 9.20) \\ &= 3.5 + 1.0714(x - 9.20), \quad 9.20 \leq x \leq 10.60 \end{aligned} \quad (5)$$

c) The length of the robot's path can be found by simply adding the length of the line segments together. The length of a straight line from one point  $(x_0, y_0)$  to another point  $(x_1, y_1)$  is given by  $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$ . Hence, the length of the linear splines from  $x = 2.00$  to  $x = 10.60$  is

$$\begin{aligned} L &= \sqrt{(4.25 - 2.00)^2 + (7.1 - 7.2)^2} + \sqrt{(5.25 - 4.25)^2 + (6.0 - 7.1)^2} \\ &\quad + \sqrt{(7.81 - 5.25)^2 + (5.0 - 6.0)^2} + \sqrt{(9.20 - 7.81)^2 + (3.5 - 5.0)^2} \\ &\quad + \sqrt{(10.60 - 9.20)^2 + (5.0 - 3.5)^2} \\ &= 10.584 \end{aligned}$$

Linear spline interpolation is no different from linear polynomial interpolation. Linear splines still use data only from the two consecutive data points. Also at the interior points of the data, the slope changes abruptly. This means that the first derivative is not continuous at these points. So how do we improve on this? We can do so by using quadratic splines.

### Quadratic Splines

In these splines, a quadratic polynomial approximates the data between two consecutive data points. Given  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , fit quadratic splines through the data.

The splines are given by

$$\begin{aligned} f(x) &= a_1x^2 + b_1x + c_1, & x_0 \leq x \leq x_1 \\ &= a_2x^2 + b_2x + c_2, & x_1 \leq x \leq x_2 \\ &\cdot \\ &\cdot \\ &= a_nx^2 + b_nx + c_n, & x_{n-1} \leq x \leq x_n \end{aligned}$$

So how does one find the coefficients of these quadratic splines? There are  $3n$  such coefficients

$$a_i, i = 1, 2, \dots, n$$

$$b_i, i = 1, 2, \dots, n$$

$$c_i, i = 1, 2, \dots, n$$

To find  $3n$  unknowns, one needs to set up  $3n$  equations and then simultaneously solve them. These  $3n$  equations are found as follows.

1. Each quadratic spline goes through two consecutive data points

$$\begin{aligned} a_1x_0^2 + b_1x_0 + c_1 &= f(x_0) \\ a_1x_1^2 + b_1x_1 + c_1 &= f(x_1) \\ &\cdot \\ &\cdot \\ &\cdot \\ a_ix_{i-1}^2 + b_ix_{i-1} + c_i &= f(x_{i-1}) \\ a_ix_i^2 + b_ix_i + c_i &= f(x_i) \\ &\cdot \\ &\cdot \\ &\cdot \\ a_nx_{n-1}^2 + b_nx_{n-1} + c_n &= f(x_{n-1}) \\ a_nx_n^2 + b_nx_n + c_n &= f(x_n) \end{aligned}$$

This condition gives  $2n$  equations as there are  $n$  quadratic splines going through two consecutive data points.

2. The first derivatives of two quadratic splines are continuous at the interior points. For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1$$

is

$$2a_1x + b_1$$

The derivative of the second spline

$$a_2x^2 + b_2x + c_2$$

is

$$2a_2x + b_2$$

and the two are equal at  $x = x_1$  giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

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$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

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$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$

Since there are  $(n-1)$  interior points, we have  $(n-1)$  such equations. So far, the total number of equations is  $(2n) + (n-1) = (3n-1)$  equations. We still then need one more equation.

We can assume that the first spline is linear, that is

$$a_1 = 0$$

This gives us  $3n$  equations and  $3n$  unknowns. These can be solved by a number of techniques used to solve simultaneous linear equations.

### Example 2

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15"×10" rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 3.

**Table 3** The coordinates of the holes on the plate.

$x$ (in.)	$y$ (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

- Find the length of the path traversed by the robot using quadratic splines.
- Compare the answer from part (a) to the linear spline result and a fifth order polynomial result.

**Solution**

a) Since there are six data points, five quadratic splines pass through them.

$$\begin{aligned} y(x) &= a_1x^2 + b_1x + c_1, & 2.00 \leq x \leq 4.25 \\ &= a_2x^2 + b_2x + c_2, & 4.25 \leq x \leq 5.25 \\ &= a_3x^2 + b_3x + c_3, & 5.25 \leq x \leq 7.81 \\ &= a_4x^2 + b_4x + c_4, & 7.81 \leq x \leq 9.20 \\ &= a_5x^2 + b_5x + c_5, & 9.20 \leq x \leq 10.60 \end{aligned}$$

The equations are found as follows.

1. Each quadratic spline passes through two consecutive data points.

$a_1x^2 + b_1x + c_1$  passes through  $x = 2.00$  and  $x = 4.25$ .

$$a_1(2.00)^2 + b_1(2.00) + c_1 = 7.2 \quad (1)$$

$$a_1(4.25)^2 + b_1(4.25) + c_1 = 7.1 \quad (2)$$

$a_2x^2 + b_2x + c_2$  passes through  $x = 4.25$  and  $x = 5.25$ .

$$a_2(4.25)^2 + b_2(4.25) + c_2 = 7.1 \quad (3)$$

$$a_2(5.25)^2 + b_2(5.25) + c_2 = 6.0 \quad (4)$$

$a_3x^2 + b_3x + c_3$  passes through  $x = 5.25$  and  $x = 7.81$ .

$$a_3(5.25)^2 + b_3(5.25) + c_3 = 6.0 \quad (5)$$

$$a_3(7.81)^2 + b_3(7.81) + c_3 = 5.0 \quad (6)$$

$a_4x^2 + b_4x + c_4$  passes through  $x = 7.81$  and  $x = 9.20$ .

$$a_4(7.81)^2 + b_4(7.81) + c_4 = 5.0 \quad (7)$$

$$a_4(9.20)^2 + b_4(9.20) + c_4 = 3.5 \quad (8)$$

$a_5x^2 + b_5x + c_5$  passes through  $x = 9.20$  and  $x = 10.60$ .

$$a_5(9.20)^2 + b_5(9.20) + c_5 = 3.5 \quad (9)$$

$$a_5(10.60)^2 + b_5(10.60) + c_5 = 5.0 \quad (10)$$

2. Quadratic splines have continuous derivatives at the interior data points.

At  $x = 4.25$

$$2a_1(4.25) + b_1 - 2a_2(4.25) - b_2 = 0 \quad (11)$$

At  $x = 5.25$

$$2a_2(5.25) + b_2 - 2a_3(5.25) - b_3 = 0 \quad (12)$$

At  $x = 7.81$

$$2a_3(7.81) + b_3 - 2a_4(7.81) - b_4 = 0 \quad (13)$$

At  $x = 9.20$

$$2a_4(9.20) + b_4 - 2a_5(9.20) - b_5 = 0 \quad (14)$$

3. Assuming the first spline  $a_1x^2 + b_1x + c_1$  is linear,

$$a_1 = 0 \quad (15)$$

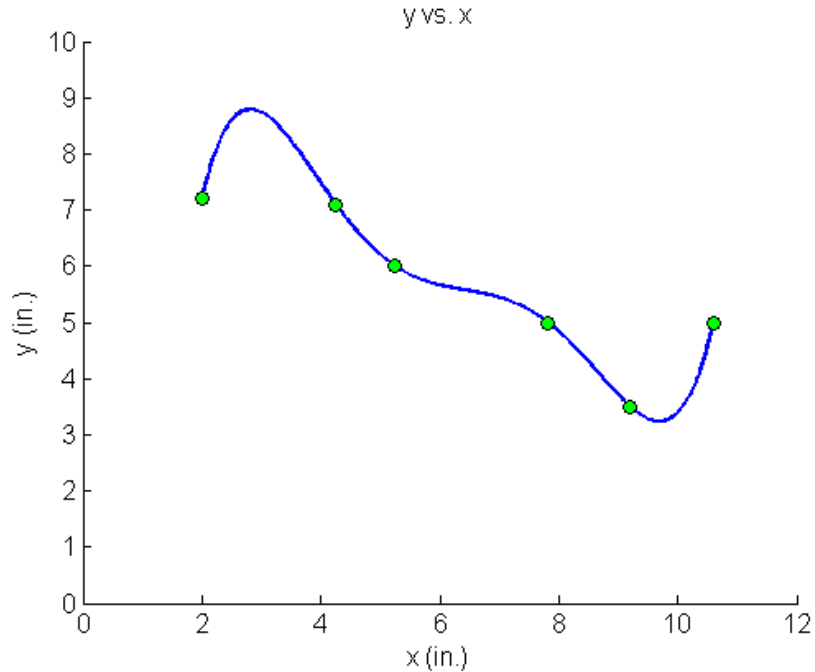
$$\begin{bmatrix} 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 18.063 & 4.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 18.063 & 4.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27.563 & 5.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 27.563 & 5.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 60.996 & 7.81 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 60.996 & 7.81 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84.64 & 9.2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84.64 & 9.2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 112.36 & 10.6 & 1 & 0 \\ 8.5 & 1 & 0 & -8.5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10.5 & 1 & 0 & -10.5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 15.62 & 1 & 0 & -15.62 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18.4 & 1 & 0 & -18.4 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix} = \begin{bmatrix} 7.2 \\ 7.1 \\ 7.1 \\ 6.0 \\ 6.0 \\ 5.0 \\ 5.0 \\ 3.5 \\ 3.5 \\ 5.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above 15 equations gives the 15 unknowns as

$i$	$a_i$	$b_i$	$c_i$
1	0	-0.044444	7.2889
2	-1.0556	8.9278	-11.777
3	0.68943	-9.3945	36.319
4	-1.7651	28.945	-113.40
5	3.2886	-64.042	314.34

Therefore, the splines are given by

$$\begin{aligned} y(x) &= -0.044444x + 7.2889, & 2.00 \leq x \leq 4.25 \\ &= -1.0556x^2 + 8.9278x - 11.777, & 4.25 \leq x \leq 5.25 \\ &= 0.68943x^2 - 9.3945x + 36.319, & 5.25 \leq x \leq 7.81 \\ &= -1.7651x^2 + 28.945x - 113.40, & 7.81 \leq x \leq 9.20 \\ &= 3.2886x^2 - 64.042x + 314.34, & 9.20 \leq x \leq 10.60 \end{aligned}$$



**Figure 6** Fifth order polynomial to traverse points of robot path (using direct method of interpolation).

The length of a function  $y = f(x)$  from  $a$  to  $b$  is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

In this case,  $f(x)$  is defined by five separate functions  $a = 2.00$  to  $b = 10.60$

$$\frac{dy}{dx} = \frac{d}{dx}(-0.044444x + 7.2889), \quad 2.00 \leq x \leq 4.25$$

$$= \frac{d}{dx}(-1.0556x^2 + 8.9278x - 11.777), \quad 4.25 \leq x \leq 5.25$$

$$= \frac{d}{dx}(0.68943x^2 - 9.3945x + 36.319), \quad 5.25 \leq x \leq 7.81$$

$$= \frac{d}{dx}(-1.7651x^2 + 28.945x - 113.40), \quad 7.81 \leq x \leq 9.20$$

$$= \frac{d}{dx}(3.2886x^2 - 64.042x + 314.34), \quad 9.20 \leq x \leq 10.60$$

$$L = \int_{2.00}^{4.25} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx + \int_{4.25}^{5.25} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx + \int_{5.25}^{7.81} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ + \int_{7.81}^{9.20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx + \int_{9.20}^{10.60} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\begin{aligned}
&= \int_{2.00}^{4.25} \sqrt{1 + (-0.04444)^2} dx + \int_{4.25}^{5.25} \sqrt{1 + (-2.1111x + 8.9278)^2} dx \\
&\quad + \int_{5.25}^{7.81} \sqrt{1 + (1.3788x - 9.3945)^2} dx + \int_{7.81}^{9.20} \sqrt{1 + (-3.5302x - 28.945)^2} dx \\
&\quad + \int_{9.20}^{10.60} \sqrt{1 + (6.5772x - 64.042)^2} dx \\
&= 2.2522 + 1.5500 + 3.6595 + 2.6065 + 3.8077 \\
&= 13.876
\end{aligned}$$

b) We can find the length of the fifth order polynomial result in a similar fashion to the quadratic splines. In this case, we do not need to break the integrals into five intervals. The fifth order polynomial result through the six points is given by

$$y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, \quad 2.00 \leq x \leq 10.6$$

Therefore,

$$\begin{aligned}
L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \int_{2.00}^{10.60} \sqrt{1 + (41.344 - 31.710x + 8.3586x^2 - 0.92364x^3 + 0.036461x^4)^2} dx \\
&= 13.123
\end{aligned}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the linear and quadratic spline is

$$\begin{aligned}
|\epsilon_a| &= \left| \frac{13.876 - 10.584}{13.876} \right| \times 100 \\
&= 0.23724\%
\end{aligned}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the fifth order polynomial and quadratic spline is

$$\begin{aligned}
|\epsilon_a| &= \left| \frac{13.876 - 13.123}{13.876} \right| \times 100 \\
&= 0.054239\%
\end{aligned}$$

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#### INTERPOLATION

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Topic	Spline Method of Interpolation
Summary	Textbook notes on the spline method of interpolation
Major	Computer Engineering
Authors	Autar Kaw, Michael Keteltas
Date	August 26, 2009
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>

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