Holistic Numerical Methods Institute

Integration

Topic: Romberg Rule

Major: Computer Engineering

Basis of Romberg Rule

Integration

The process of measuring the area under a curve.

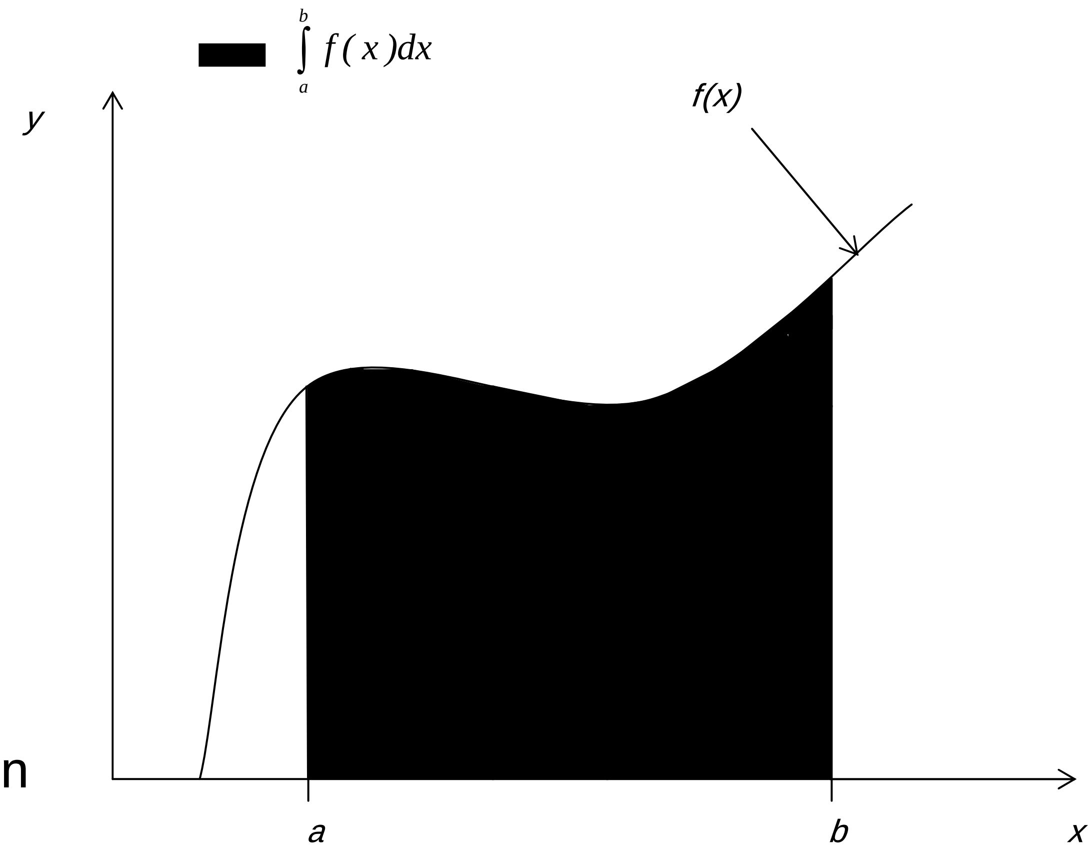
$$I = \int_a^b f(x)dx$$

Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration



What is The Romberg Rule?

Romberg Integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integral by reducing the True Error.

Error in Multiple Segment Trapezoidal Rule

The true error in a multiple segment Trapezoidal Rule with n segments for an integral

$$I = \int_a^b f(x)dx$$

Is given by

$$E_t = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

where for each *i*, ξ_i is a point somewhere in the domain , $[a+(i-1)h, a+ih]$.

Error in Multiple Segment Trapezoidal Rule

The term $\dfrac{\sum_{i=1}^{n} f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$.

This leads us to say that the true error, E_t previously defined can be approximated as

$$E_t \cong \alpha \frac{1}{n^2}$$

Error in Multiple Segment Trapezoidal Rule

Table 1 shows the results obtained for the integral using multiple segment Trapezoidal rule for

$$f(x) = 0, \; 0 < x < 30$$
$$= -9.1688\times 10^{-6}x^3 + 2.7961\times 10^{-3}x^2 - 2.8487\times 10^{-1}x + 9.6778, \quad 30 \le x \le 172$$
$$= 0, \quad 172 < x < 200$$

n	Value	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
1	-0.85	61.6426	101.398	---
2	63.4975	-2.7049	4.44939	101.338
3	111.257	-50.4648	83.0114	42.927
4	36.061	24.73073	40.68049	208.518
5	58.427	2.36519	3.8906	38.279
6	77.769	-16.9765	27.9253	24.870
7	42.527	18.2648	30.0444	82.866
8	55.753	5.03879	8.2885	23.722

Table 1: Multiple Segment Trapezoidal Rule Values

Error in Multiple Segment Trapezoidal Rule

The true error gets approximately quartered as the number of segments is doubled. This information is used to get a better approximation of the integral, and is the basis of Richardson's extrapolation.

Richardson's Extrapolation for Trapezoidal Rule

The true error, E_t in the *n*-segment Trapezoidal rule is estimated as

$$E_t \cong \frac{C}{n^2}$$

where *C* is an *approximate constant* of proportionality. Since

$$E_t = TV - I_n$$

Where TV = true value and I_n = approx. value

Richardson's Extrapolation for Trapezoidal Rule

From the previous development, it can be shown that

$$\frac{C}{(2n)^2} \cong TV - I_{2n}$$

when the segment size is doubled and that

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

which is Richardson's Extrapolation.

Example 1

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to integrated.

$$I = \int_{0}^{100} f(x)dx$$

where

$$\begin{aligned} f(x) &= 0, \; 0 < x < 30 \\ &= -9.1688\times10^{-6}x^3 + 2.7961\times10^{-3}x^2 - 2.8487\times10^{-1}x + 9.6778, \\ &\quad 30 \le x \le 172 \\ &= 0, \; 172 < x < 200 \end{aligned}$$

a) Use Richardson's rule to find the distance covered. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.

b) Find the true error, E_t for part (a).

c) Find the absolute relative true error, $|\varepsilon_a|$ for part (a).

Solution

a) $I_2 = 63.4975$ $I_4 = 36.061$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing *n=2*,

$$TV \cong I_4 + \frac{I_4 - I_2}{3} \qquad = 63.4975 + \frac{63.4975 - 36.061}{3}$$

$$= 72.643$$

Solution (cont.)

b) The exact value of the above integral is found using Maple for calculating the true error and relative true error.

$$I = \int_{0}^{100} f(x)dx$$

$$= 60.7926$$

Hence

$$E_t = True\ Value - Approximate\ Value$$

$$= 60.7926 - 72.643$$

$$= -11.8504$$

Solution (cont.)

c) The absolute relative true error $\left|\in_t\right|$ would then be

$$\left|\in_t\right| = \left|\frac{60.7926 - 72.643}{60.7926}\right| \times 100$$

$$= 19.49\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Solution (cont.)

Table 2: The values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$f(x) = 0,\ 0 < x < 30$$
$$= -9.1688\times10^{-6}x^3 + 2.7961\times10^{-3}x^2 - 2.8487\times10^{-1}x + 9.6778,\ \ 30 \le x \le 172$$
$$= 0,\ \ 172 < x < 200$$

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n	Trapezoidal Rule	$\lvert\in_t\rvert$ for Trapezoidal Rule	Richardson's Extrapolation	$\lvert\in_t\rvert$ for Richardson's Extrapolation
1	-0.85	101.398	--	--
2	63.4975	4.44939	84.947	39.732
4	36.061	40.68049	72.643	19.49
8	55.753	8.2885	62.317	2.508

Table 2: Richardson's Extrapolation Values

Romberg Integration

Romberg integration is same as Richardson's extrapolation formula as given previously. However, Romberg used a recursive algorithm for the extrapolation. Recall

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

This can alternately be written as

$$\left(I_{2n}\right)_R = I_{2n} + \frac{I_{2n} - I_n}{3} \quad = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$

Romberg Integration

Note that the variable TV is replaced by $\left(I_{2n}\right)_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign $\cong$ is replaced by = sign.

Hence the estimate of the true value now is

$$TV \cong \left(I_{2n}\right)_R + Ch^4$$

Where Ch^4 is an approximation of the true error.

Romberg Integration

Determine another integral value with further halving the step size (doubling the number of segments),

$$\left(I_{4n}\right)_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

It follows from the two previous expressions that the true value TV can be written as

$$TV \cong \left(I_{4n}\right)_R + \frac{\left(I_{4n}\right)_R - \left(I_{2n}\right)_R}{15}$$

$$= I_{4n} + \frac{\left(I_{4n}\right)_R - \left(I_{2n}\right)_R}{4^{3-1} - 1}$$

Romberg Integration

A general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index *k* represents the order of extrapolation. k=1 represents the values obtained from the regular Trapezoidal rule, k=2 represents values obtained using the true estimate as O(h^2). The index *j* represents the more and less accurate estimate of the integral.

Example 2

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to integrated.

$$I = \int_{0}^{100} f(x)dx$$

where

$$f(x) = 0,\ 0 < x < 30$$

$$= -9.1688\times10^{-6}x^3 + 2.7961\times10^{-3}x^2 - 2.8487\times10^{-1}x + 9.6778,$$

$$30 \le x \le 172$$

$$= 0,\ 172 < x < 200$$

Use Romberg's rule to find the distance covered. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given in the Table 1.

Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = -0.85 \qquad I_{1,2} = 63.4975$$

$$I_{1,3} = 36.061 \qquad I_{1,4} = 55.753$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively.

Solution (cont.)

To get the first order extrapolation values,

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$

$$= 63.4975 + \frac{63.4975 - (-0.85)}{3}$$

$$= 84.947$$

Similarly,

$$I_{2,2} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3}$$

$$= 36.061 + \frac{36.061 - 63.4975}{3}$$

$$= 26.917$$

$$I_{2,3} = I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3}$$

$$= 55.753 + \frac{55.753 - 36.061}{3}$$

$$= 62.318$$

Solution (cont.)

For the second order extrapolation values,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

$$= 26.917 + \frac{26.917 - 84.947}{15}$$

$$= 23.048$$

Similarly,

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$

$$= 62.318 + \frac{62.318 - 26.917}{15}$$

$$= 64.678$$

Solution (cont.)

For the third order extrapolation values,

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$

$$= 64.678 + \frac{64.678 - 23.048}{63}$$

$$= 65.338$$

Table 3 shows these increased correct values in a tree graph.

Solution (cont.)

Table 3: Improved estimates of the integral value using Romberg Integration

		1^{st} Order	2^{nd} Order	3^{rd} Order
1-segment	-0.85			
		84.947		
			23.048	
2-segment	63.4975			
				65.338
		26.917		
			64.678	
4-segment	36.061			
		62.318		
8-segment	55.753			