

Chapter 07.03

Simpson's 1/3 Rule of Integration

After reading this chapter, you should be able to

1. *derive the formula for Simpson's 1/3 rule of integration,*
2. *use Simpson's 1/3 rule to solve integrals,*
3. *develop the formula for multiple-segment Simpson's 1/3 rule of integration,*
4. *use multiple-segment Simpson's 1/3 rule of integration to solve integrals, and*
5. *derive the true error formula for multiple-segment Simpson's 1/3 rule.*

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral. Here, we will discuss Simpson's 1/3 rule of integral approximation, which improves upon the accuracy of the trapezoidal rule.

Here, we will discuss the Simpson's 1/3 rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

$f(x)$ is called the integrand,

a = lower limit of integration

b = upper limit of integration

Simpson's 1/3 Rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's 1/3 rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

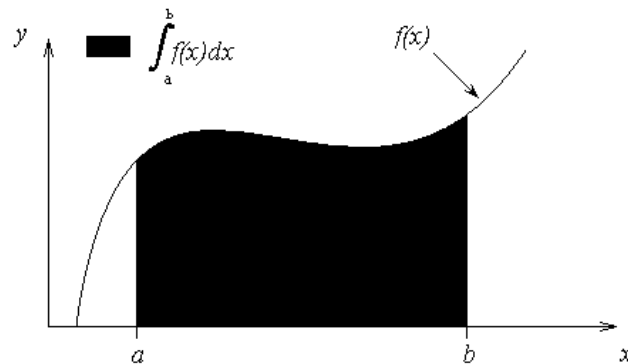


Figure 1 Integration of a function

Method 1:

Hence

$$I = \int_a^b f(x) dx \approx \int_a^b f_2(x) dx$$

where $f_2(x)$ is a second order polynomial given by

$$f_2(x) = a_0 + a_1x + a_2x^2$$

Choose

$$(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right) \right), \text{ and } (b, f(b))$$

as the three points of the function to evaluate a_0 , a_1 and a_2 .

$$f(a) = f_2(a) = a_0 + a_1a + a_2a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1b + a_2b^2$$

Solving the above three equations for unknowns, a_0 , a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$\begin{aligned}
 I &\approx \int_a^b f_2(x) dx \\
 &= \int_a^b (a_0 + a_1 x + a_2 x^2) dx \\
 &= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} \right]_a^b \\
 &= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}
 \end{aligned}$$

Substituting values of a_0 , a_1 and a_2 give

$$\int_a^b f_2(x) dx = \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Since for Simpson 1/3 rule, the interval $[a, b]$ is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Hence the Simpson's 1/3 rule is given by

$$\int_a^b f(x) dx \approx \frac{h}{3} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Since the above form has 1/3 in its formula, it is called Simpson's 1/3 rule.

Method 2:

Simpson's 1/3 rule can also be derived by approximating $f(x)$ by a second order polynomial using Newton's divided difference polynomial as

$$f_2(x) = b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)$$

where

$$b_0 = f(a)$$

$$b_1 = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}$$

$$b_2 = \frac{\frac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} - \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}}{b-a}$$

Integrating Newton's divided difference polynomial gives us

$$\begin{aligned}
\int_a^b f(x) dx &\approx \int_a^b f_2(x) dx \\
&= \int_a^b \left[b_0 + b_1(x-a) + b_2(x-a) \left(x - \frac{a+b}{2} \right) \right] dx \\
&= \left[b_0 x + b_1 \left(\frac{x^2}{2} - ax \right) + b_2 \left(\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2} \right) \right]_a^b \\
&= b_0(b-a) + b_1 \left(\frac{b^2 - a^2}{2} - a(b-a) \right) \\
&\quad + b_2 \left(\frac{b^3 - a^3}{3} - \frac{(3a+b)(b^2 - a^2)}{4} + \frac{a(a+b)(b-a)}{2} \right)
\end{aligned}$$

Substituting values of b_0 , b_1 , and b_2 into this equation yields the same result as before

$$\begin{aligned}
\int_a^b f(x) dx &\approx \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\
&= \frac{h}{3} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\end{aligned}$$

Method 3:

One could even use the Lagrange polynomial to derive Simpson's formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes

$$f_2(x) = \frac{\left(x - \frac{a+b}{2}\right)(x-b)}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) + \frac{(x-a)(x-b)}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) + \frac{(x-a)\left(x - \frac{a+b}{2}\right)}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b)$$

Integrating this function gets

$$\int_a^b f_2(x) dx = \left[\frac{\frac{x^3}{3} - \frac{(a+3b)x^2}{4} + \frac{b(a+b)x}{2}}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) + \frac{\frac{x^3}{3} - \frac{(a+b)x^2}{2} + abx}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) + \frac{\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b) \right]_a^b$$

$$\begin{aligned}
&= \frac{\frac{b^3 - a^3}{3} - \frac{(a+3b)(b^2 - a^2)}{4} + \frac{b(a+b)(b-a)}{2}}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) \\
&+ \frac{\frac{b^3 - a^3}{3} - \frac{(a+b)(b^2 - a^2)}{2} + ab(b-a)}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) \\
&+ \frac{\frac{b^3 - a^3}{3} - \frac{(3a+b)(b^2 - a^2)}{4} + \frac{a(a+b)(b-a)}{2}}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b)
\end{aligned}$$

Believe it or not, simplifying and factoring this large expression yields you the same result as before

$$\begin{aligned}
\int_a^b f(x) dx &\approx \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\
&= \frac{h}{3} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right].
\end{aligned}$$

Method 4:

Simpson's 1/3 rule can also be derived by the method of coefficients. Assume

$$\int_a^b f(x) dx \approx c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$

Let the right-hand side be an exact expression for the integrals $\int_a^b 1 dx$, $\int_a^b x dx$, and $\int_a^b x^2 dx$. This implies that the right hand side will be exact expressions for integrals of any linear combination of the three integrals for a general second order polynomial. Now

$$\begin{aligned}
\int_a^b 1 dx &= b - a = c_1 + c_2 + c_3 \\
\int_a^b x dx &= \frac{b^2 - a^2}{2} = c_1 a + c_2 \frac{a+b}{2} + c_3 b \\
\int_a^b x^2 dx &= \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 \left(\frac{a+b}{2}\right)^2 + c_3 b^2
\end{aligned}$$

Solving the above three equations for c_0 , c_1 and c_2 give

$$c_1 = \frac{b-a}{6}$$

$$c_2 = \frac{2(b-a)}{3}$$

$$c_3 = \frac{b-a}{6}$$

This gives

$$\begin{aligned} \int_a^b f(x)dx &\approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b) \\ &= \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\ &= \frac{h}{3} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \end{aligned}$$

The integral from the first method

$$\int_a^b f(x)dx \approx \int_a^b (a_0 + a_1x + a_2x^2)dx$$

can be viewed as the area under the second order polynomial, while the equation from Method 4

$$\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$

can be viewed as the sum of the areas of three rectangles.

Example 1

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to be integrated.

$$I = \int_0^{100} f(x)dx$$

where

$$\begin{aligned} f(x) &= 0, \quad 0 < x < 30 \\ &= -9.1688 \times 10^{-6} x^3 + 2.7961 \times 10^{-3} x^2 - 2.8487 \times 10^{-1} x + 9.6778, \quad 30 \leq x \leq 172 \\ &= 0, \quad 172 < x < 200 \end{aligned}$$

- Use Simpson's 1/3 Rule to find the probability.
- Find the true error, E_t , for part (a).
- Find the absolute relative true error for part (a).

Solution

$$\begin{aligned} \text{a) } I &\approx \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\ a &= 0 \end{aligned}$$

$$b = 100$$

$$\frac{a+b}{2} = 50$$

$$f(x) = 0, \quad 0 < x < 30$$

$$= -9.1688 \times 10^{-6} x^3 + 2.7961 \times 10^{-3} x^2 - 2.8487 \times 10^{-1} x + 9.6778, \quad 30 \leq x \leq 172$$

$$= 0, \quad 172 < x < 200$$

$$f(0) = 0$$

$$f(100) = -9.1688 \times 10^{-6} \times (100)^3 + 2.7961 \times 10^{-3} \times (100)^2 - 2.8487 \times 10^{-1} \times (100) + 9.6778$$

$$= -0.017000$$

$$f(50) = -9.1688 \times 10^{-6} \times (50)^3 + 2.7961 \times 10^{-3} \times (50)^2 - 2.8487 \times 10^{-1} \times (50) + 9.6778$$

$$= 1.2784$$

$$I \approx \left(\frac{b-a}{6} \right) \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$\approx \left(\frac{100-0}{6} \right) [f(0) + 4f(50) + f(100)]$$

$$\approx \left(\frac{100}{6} \right) [0 + 4(1.2784) + (-0.017)]$$

$$\approx 84.947$$

b) The exact value of the above integral is found using Maple for calculating the true error and relative true error.

$$I = \int_0^{100} f(x) dx$$

$$= 60.793$$

so the true error is

$$E_t = \text{True Value} - \text{Approximate Value}$$

$$= 60.793 - (84.947)$$

$$= -24.154$$

c) The absolute relative true error, $|\epsilon_t|$, would then be

$$|\epsilon_t| = \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \%$$

$$= \left| \frac{60.793 - (84.947)}{60.793} \right| \times 100 \%$$

$$= 39.732 \%$$

Multiple-segment Simpson's 1/3 Rule

Just like in multiple-segment trapezoidal rule, one can subdivide the interval $[a, b]$ into n segments and apply Simpson's 1/3 rule repeatedly over every two segments. Note that n

needs to be even. Divide interval $[a, b]$ into n equal segments, so that the segment width is given by

$$h = \frac{b-a}{n}.$$

Now

$$\int_a^b f(x) dx = \int_{x_0}^{x_n} f(x) dx$$

where

$$x_0 = a$$

$$x_n = b$$

$$\int_a^b f(x) dx = \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \dots + \int_{x_{n-4}}^{x_{n-2}} f(x) dx + \int_{x_{n-2}}^{x_n} f(x) dx$$

Apply Simpson's 1/3 Rule over each interval,

$$\int_a^b f(x) dx \cong (x_2 - x_0) \left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + (x_4 - x_2) \left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \dots$$

$$+ (x_{n-2} - x_{n-4}) \left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + (x_n - x_{n-2}) \left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right]$$

Since

$$x_i - x_{i-2} = 2h$$

$$i = 2, 4, \dots, n$$

then

$$\int_a^b f(x) dx \cong 2h \left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + 2h \left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \dots$$

$$+ 2h \left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + 2h \left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right]$$

$$= \frac{h}{3} [f(x_0) + 4\{f(x_1) + f(x_3) + \dots + f(x_{n-1})\} + 2\{f(x_2) + f(x_4) + \dots + f(x_{n-2})\} + f(x_n)]$$

$$= \frac{h}{3} \left[f(x_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n) \right]$$

$$\int_a^b f(x) dx \cong \frac{b-a}{3n} \left[f(x_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n) \right]$$

Example 2

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to be integrated.

$$I = \int_0^{100} f(x) dx$$

where

$$f(x) = 0, \quad 0 < x < 30$$

$$= -9.1688 \times 10^{-6} x^3 + 2.7961 \times 10^{-3} x^2 - 2.8487 \times 10^{-1} x + 9.6778, \quad 30 \leq x \leq 172$$

$$= 0, \quad 172 < x < 200$$

- Use four segment Simpson's 1/3 Rule to find the value of the integral
- Find the true error, E_t , for part (a).
- Find the absolute relative true error for part (a).

Solution

a) Using n segment Simpson's 1/3 Rule,

$$I \approx \frac{b-a}{3n} \left[f(x_0) + 4 \sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

$$n = 4$$

$$a = 0$$

$$b = 100$$

$$h = \frac{b-a}{n}$$

$$= \frac{100-0}{4}$$

$$= 25$$

$$f(x) = 0, \quad 0 < x < 30$$

$$= -9.1688 \times 10^{-6} x^3 + 2.7961 \times 10^{-3} x^2 - 2.8487 \times 10^{-1} x + 9.6778, \quad 30 \leq x \leq 172$$

$$= 0, \quad 172 < x < 200$$

$$I \approx \frac{b-a}{3n} \left[f(x_0) + 4 \sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

$$\approx \frac{100-0}{3(4)} \left[f(0) + 4 \sum_{\substack{i=1 \\ i=odd}}^3 f(x_i) + 2 \sum_{\substack{i=2 \\ i=even}}^2 f(x_i) + f(100) \right]$$

$$\approx \frac{100}{12} [f(0) + 4f(x_1) + 4f(x_3) + 2f(x_2) + f(100)]$$

$$\begin{aligned}
&\approx \frac{25}{3} [f(0) + 4f(25) + 4f(75) + 2f(50) + f(100)] \\
&\approx \frac{25}{3} [0 + 4(0) + 4(0.17253) + 2(1.2784) + (-0.017000)] \\
&\approx 26.917
\end{aligned}$$

Since

$$\begin{aligned}
f(x_0) &= f(0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
f(x_1) &= f(0 + 25) \\
&= f(25) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
f(x_2) &= f(25 + 25) \\
&= f(50) \\
&= -9.1688 \times 10^{-6} \times (50)^3 + 2.7961 \times 10^{-3} \times (50)^2 - 2.8487 \times 10^{-1} \times (50) + 9.6778 \\
&= 1.2784
\end{aligned}$$

$$\begin{aligned}
f(x_3) &= f(50 + 25) \\
&= f(75) \\
&= -9.1688 \times 10^{-6} \times (75)^3 + 2.7961 \times 10^{-3} \times (75)^2 - 2.8487 \times 10^{-1} \times (75) + 9.6778 \\
&= 0.17253
\end{aligned}$$

$$\begin{aligned}
f(x_4) &= f(x_n) \\
&= f(100) \\
&= -9.1688 \times 10^{-6} \times (100)^3 + 2.7961 \times 10^{-3} \times (100)^2 - 2.8487 \times 10^{-1} \times (100) + 9.6778 \\
&= 0.017000
\end{aligned}$$

b) The exact value of the above integral is found using Maple for calculating the true error and relative true error.

$$\begin{aligned}
I &= \int_0^{100} f(x) dx \\
&= 60.793
\end{aligned}$$

so the true error is

$$\begin{aligned}
E_t &= \text{True Value} - \text{Approximate Value} \\
&= 60.793 - (26.917) \\
&= 33.873
\end{aligned}$$

c) The absolute relative true error, $|\epsilon_t|$, would then be

$$\begin{aligned}
 |\epsilon_t| &= \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \% \\
 &= \left| \frac{60.793 - (26.917)}{60.793} \right| \times 100 \% \\
 &= 55.724 \%
 \end{aligned}$$

Table 1 Values of Simpson's 1/3 Rule for Example 2 with multiple segments.

n	Approximate Value	E_t	$ \epsilon_t $ %
2	84.947	-24.154	39.732
4	26.917	33.876	55.724
6	66.606	-5.8138	9.5633
8	62.318	-1.5252	2.5088
10	85.820	-25.023	41.169

Error in Multiple-segment Simpson's 1/3 rule

The true error in a single application of Simpson's 1/3 Rule is given¹ by

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In multiple-segment Simpson's 1/3 rule, the error is the sum of the errors in each application of Simpson's 1/3 rule. The error in the n segments Simpson's 1/3 Rule is given by

$$\begin{aligned}
 E_1 &= -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2 \\
 &= -\frac{h^5}{90} f^{(4)}(\zeta_1) \\
 E_2 &= -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4 \\
 &= -\frac{h^5}{90} f^{(4)}(\zeta_2) \\
 &\vdots \\
 E_i &= -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i} \\
 &= -\frac{h^5}{90} f^{(4)}(\zeta_i) \\
 &\vdots
 \end{aligned}$$

¹ The $f^{(4)}$ in the true error expression stands for the fourth derivative of the function $f(x)$.

$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}$$

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right)$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), \quad x_{n-2} < \zeta_{\frac{n}{2}} < x_n$$

Hence, the total error in the multiple-segment Simpson's 1/3 rule is

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right)$$

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i$$

$$= -\frac{h^5}{90} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^5} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^4} \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

The term $\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$ is an approximate average value of $f^{(4)}(x)$, $a < x < b$. Hence

$$E_t = -\frac{(b-a)^5}{90n^4} \bar{f}^{(4)}$$

where

$$\bar{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

INTEGRATION

Topic	Simpson's 1/3 rule
Summary	Textbook notes of Simpson's 1/3 rule
Major	Computer Engineering
Authors	Autar Kaw, Michael Keteltas
Date	August 27, 2009
Web Site	http://numericalmethods.eng.usf.edu
