# Ordinary Differential Equations

## Topic: Euler Method

Major: Electrical Engineering

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# Euler's Method

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

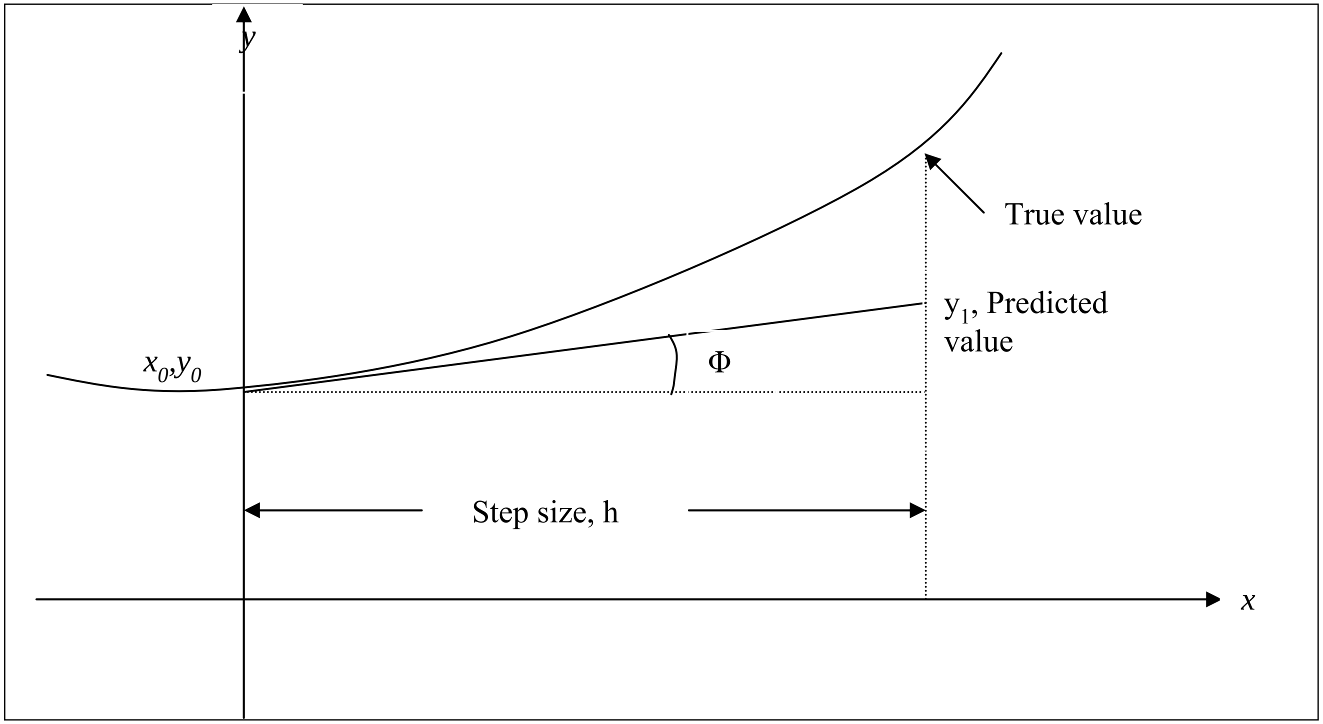


**Figure 1.** Graphical interpretation of the first step of Euler's method

# Euler's Method

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h$$

$$h = x_{i+1} - x_i$$

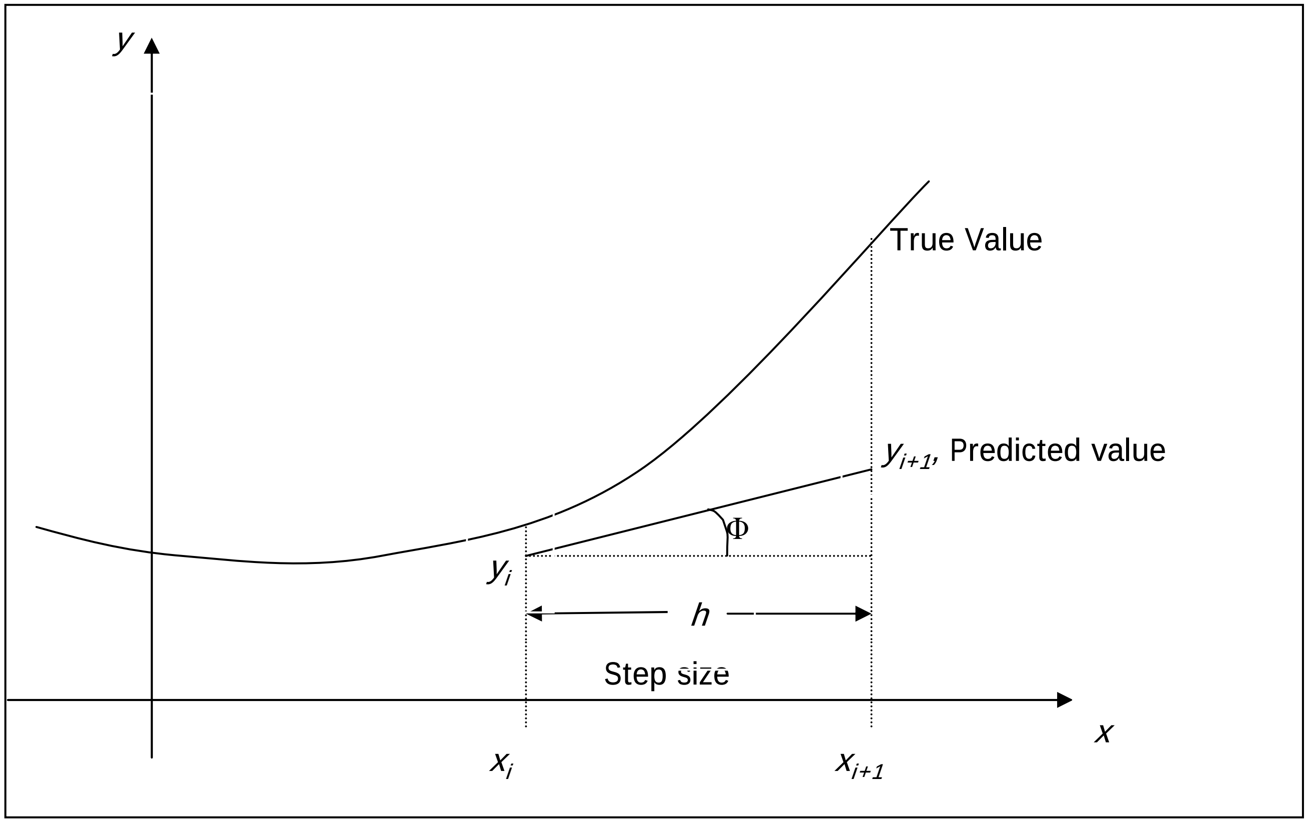


**Figure 2.** General graphical interpretation of Euler's method

# How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

**Example**

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

# Example

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of 150 µF, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt} = \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(t))|-2-v(t)}{0.04},0\right)\right\} \qquad v(0)=0$$

Find voltage across the capacitor at t= 0.00004s. Use step size h=0.00002

$$\frac{dv}{dt} = \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(t))|-2-v}{0.04},0\right)\right\}$$

$$f(t,v) = \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(t))|-2-v}{0.04},0\right)\right\}$$

$$v_{i+1} = v_i + f(t_i, v_i)h$$

# Solution

Step 1: $v_1 = v_0 + f(t_0, v_0)h$

$$= 0 + f(0,0)0.00002$$

$$= \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0))| - 2 - (0)}{0.04}, 0\right)\right\}0.00002$$

$$= 0 + \left(2.6667\times 10^{6}\right)0.00002$$

$$= 53.322V$$

$v_1$ is the approximate voltage at

$$t = t_1 = t_0 + h = 0 + 0.00002 = 0.00002$$

$$v_1 = v(0.00002) \cong 53.322V$$

# Solution Cont

**Step 2:**

$$v_2 = v_1 + f\left(t_1, v_1\right)h$$
$$= 53.322 + f\left(0.00002, 53.322\right)0.00002$$

$$= 53.322 + \frac{1}{150\times10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00002))\right| - 2 - (53.322)}{0.04}, 0\right)\right\}0.00002$$

$$= 53.322 + \left(-0.000015000\right)0.00002$$

$$= 53.322V$$

$v_2$ is the approximate voltage at

$$t = t_2 = t_1 + h = 0.00002 + 0.00002 = 0.00004s$$

$$v_2 = v\left(0.00004\right) \cong 53.322V$$

# Solution Cont

The solution to this nonlinear equation at t=0.00004 seconds is

$$v(0.00004) = 15.974V$$

# Comparison of Exact and Numerical Solutions

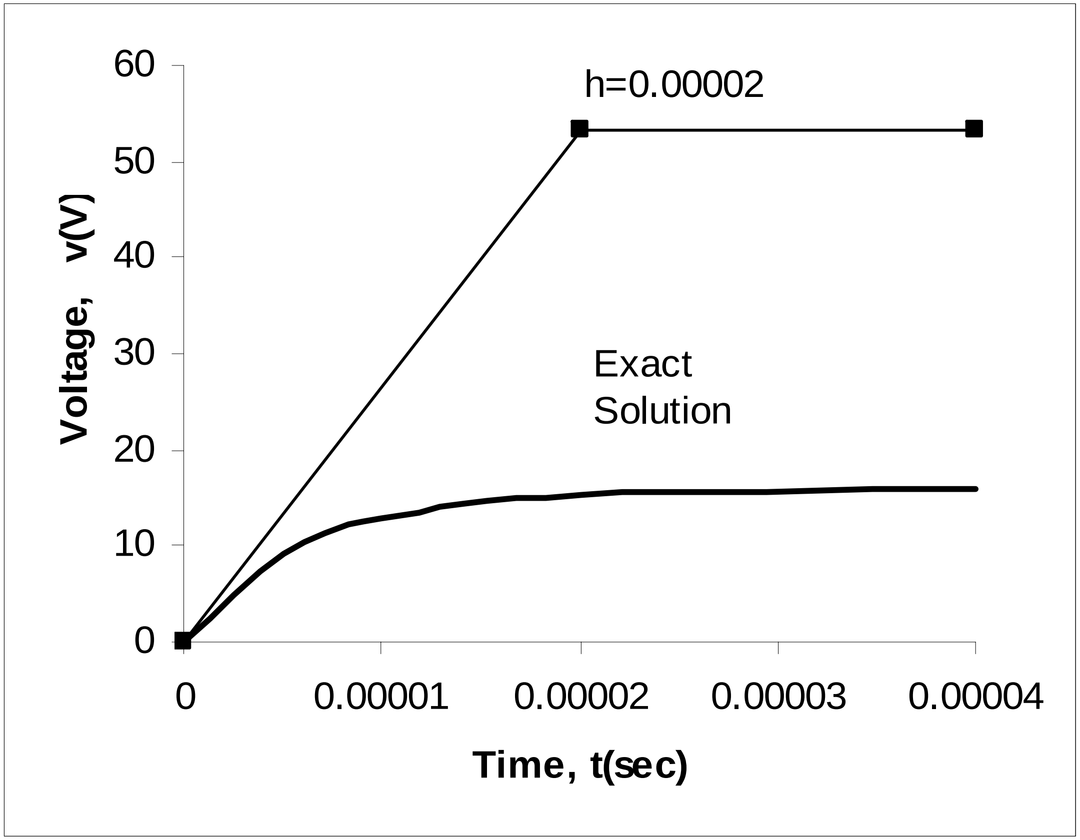


**Figure 3.** Comparing exact and Euler's method

# Effect of step size

**Table 1. Voltage at 0.00004 seconds as a function of step size, h**

| Step $h$ | $v(0.00004)$ | $E_t$ | $\lvert\in_t\rvert$ % |
|---|---|---|---|
| 0.00004 | 106.64 | -90.667 | 567.59 |
| 0.00002 | 53.307 | -37.333 | 233.71 |
| 0.00001 | 26.640 | -10.666 | 66.771 |
| 0.000005 | 15.995 | -0.021000 | 0.13146 |
| 0.0000025 | 15.992 | -0.018000 | 0.11268 |

# Comparison with exact results

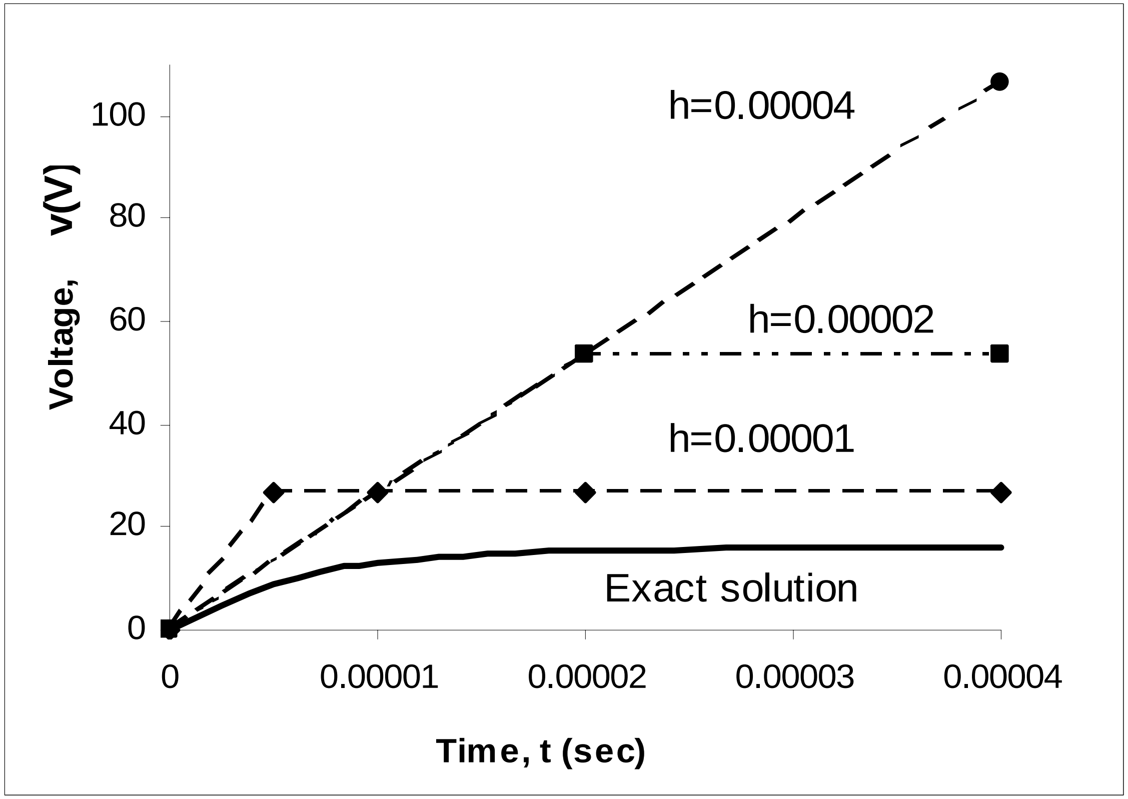


**Figure 4.** Comparison of Euler's method with exact solution for different step sizes

# Effects of step size on Euler's Method

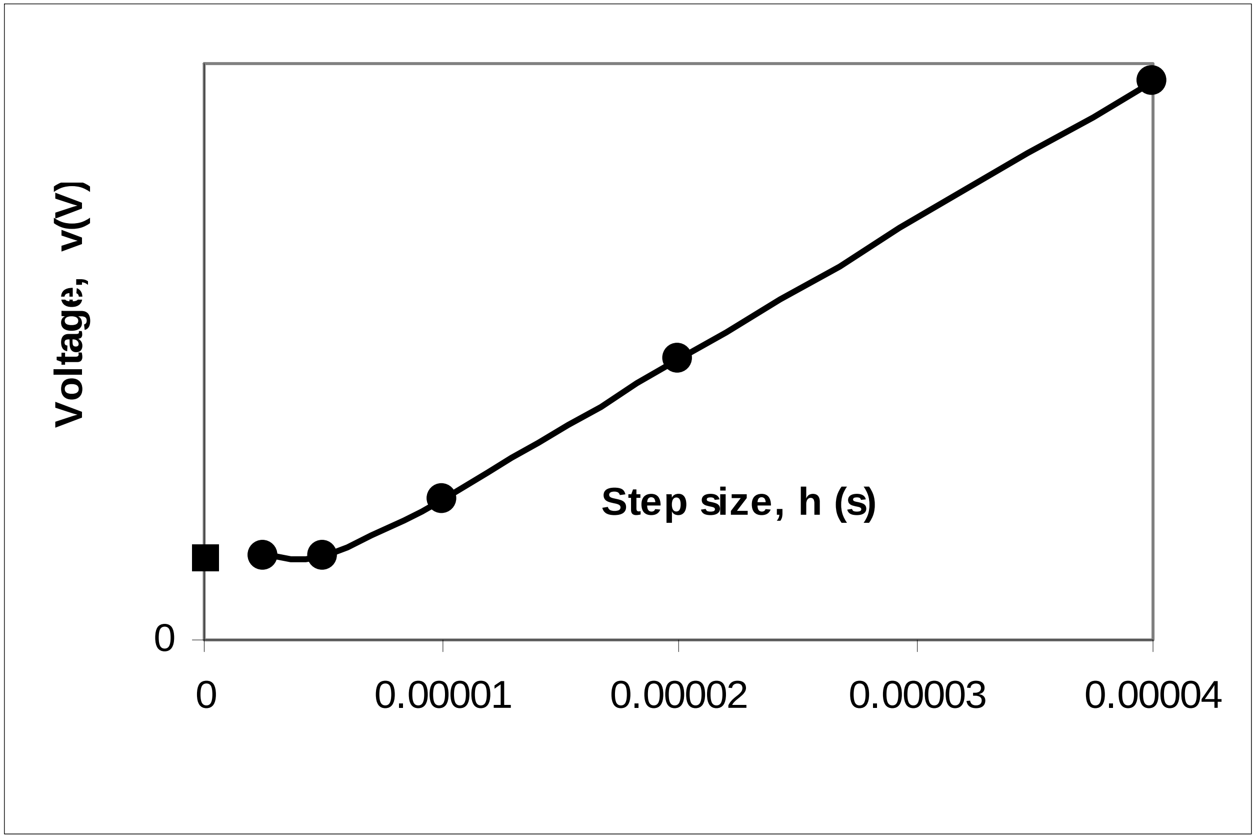


**Figure 5.** Effect of step size in Euler's method.

# Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i,y_i}(x_{i+1}-x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i,y_i}(x_{i+1}-x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i,y_i}(x_{i+1}-x_i)^3 + \ldots$$

$$y_{i+1} = y_i + f(x_i,y_i) + \frac{1}{2!}f'(x_i,y_i)(x_{i+1}-x_i)^2 + \frac{1}{3!}f''(x_i,y_i)(x_{i+1}-x_i)^3 + \ldots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i,y_i)h$ are the Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i,y_i)}{2!}h^2 + \frac{f''(x_i,y_i)}{3!}h^3 + \ldots \qquad E_t \propto h^2$$