Ordinary Differential Equations

Topic: Runge-Kutta 2nd Order Method

Major: Electrical Engineering

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Runge-Kutta 2nd Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + p_1 h, y_i + q_{11} k_1 h\right)$$

Heun's Method

<u>Heun's method</u>

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

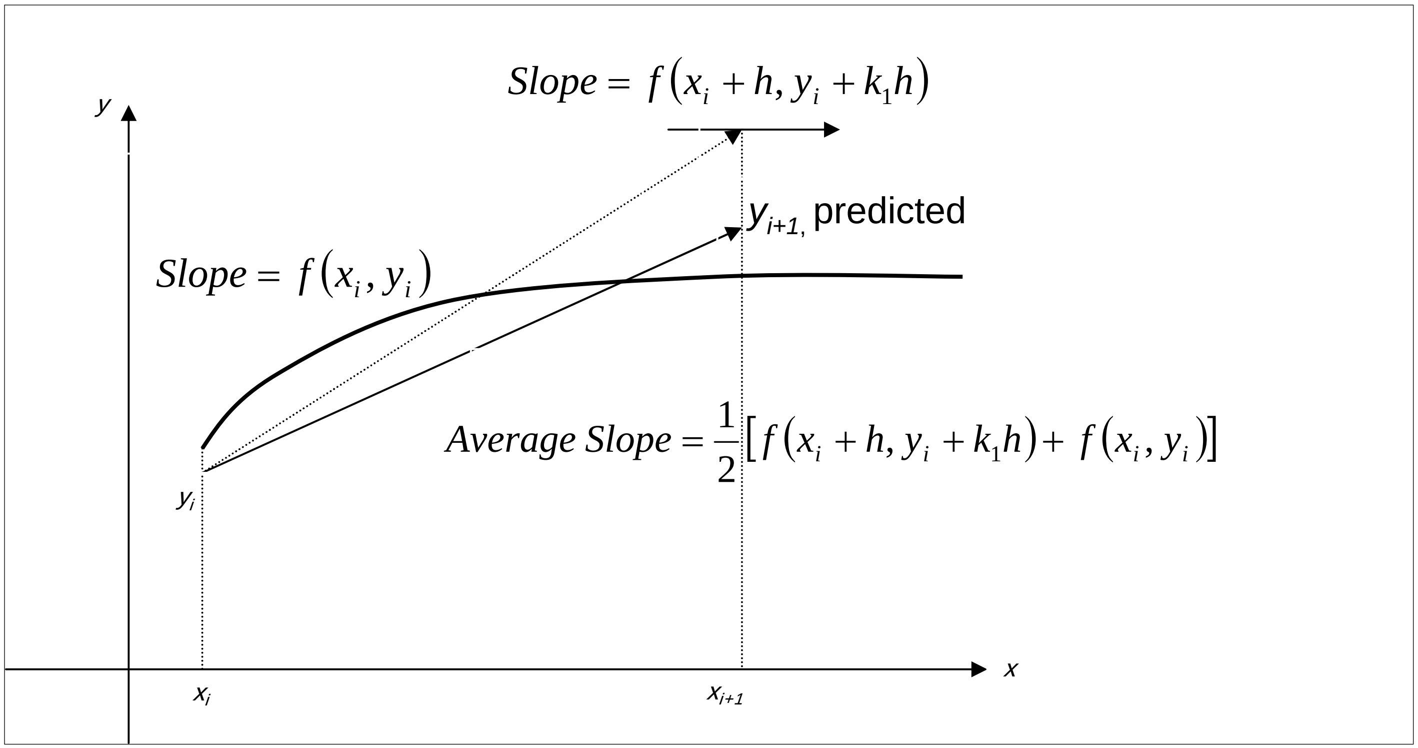

Figure 1. Runge-Kutta 2nd order method (Heun's method)

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + (\frac{1}{3}k_1 + \frac{2}{3}k_2)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of 150 µF, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt}=\frac{1}{150\times 10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v(t)}{0.04},0\right)\right\} \qquad v(0)=0$$

Find voltage across the capacitor at t= 0.00004s. Use step size h=0.00002

$$\frac{dv}{dt}=\frac{1}{150\times 10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v}{0.04},0\right)\right\}$$

$$f(t,v)=\frac{1}{150\times 10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v}{0.04},0\right)\right\}$$

$$v_{i+1}=v_i+\left(\frac{1}{2}k_1+\frac{1}{2}k_2\right)h$$

Solution

Step 1: $i=0, t_0=0, v_0=v(0)=0$

$$k_1 = f(t_0, v_o) = f(0,0) = \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0))|-2-(0)}{0.04},0\right)\right\}$$

$$= 2.6667\times10^6$$

$$k_2 = f(t_0+h, v_0+k_1h) = f\left(0+0.00002, 0+\left(2.6667\times10^6\right)0.00002\right) = f(0.00002, 53.32)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0.00002))|-2-(53.32)}{0.04},0\right)\right\} = -666.67$$

$$v_1 = v_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h = 0 + \left(\frac{1}{2}\left(2.6667\times10^6\right) + \frac{1}{2}(-666.67)\right)0.00002$$

$$= 0 + \left(1.3330\times10^6\right)0.00002 = 26.660V$$

Solution Cont

Step 2: $i=1, t_1 = t_0 + h = 0 + 0.00002 = 0.00002 \quad v_1 = 26.660V$

$$k_1 = f(t_1, v_1) = f(0.00002, 26.660) = \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00002))\right| - 2 - (26.660)}{0.04}, 0\right)\right\}$$

$$= -666.67$$

$$k_2 = f(t_1 + h, v_1 + k_1 h) = f(0.00002 + 0.00002, 26.660 + (-666.67)0.00002) = f(0.00004, 26.647)$$

$$= \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00004))\right| - 2 - (26.647)}{0.04}, 0\right)\right\} = -666.67$$

$$v_2 = v_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h = 26.660 + \left(\frac{1}{2}(-666.67) + \frac{1}{2}(-666.67)\right)0.00002$$

$$= 26.660 + (-666.67)0.00002 = 26.647V$$

Solution Continued

The solution to this nonlinear equation at t=0.00004 seconds is

$$v(0.00004) = 15.974V$$

Comparison with exact results

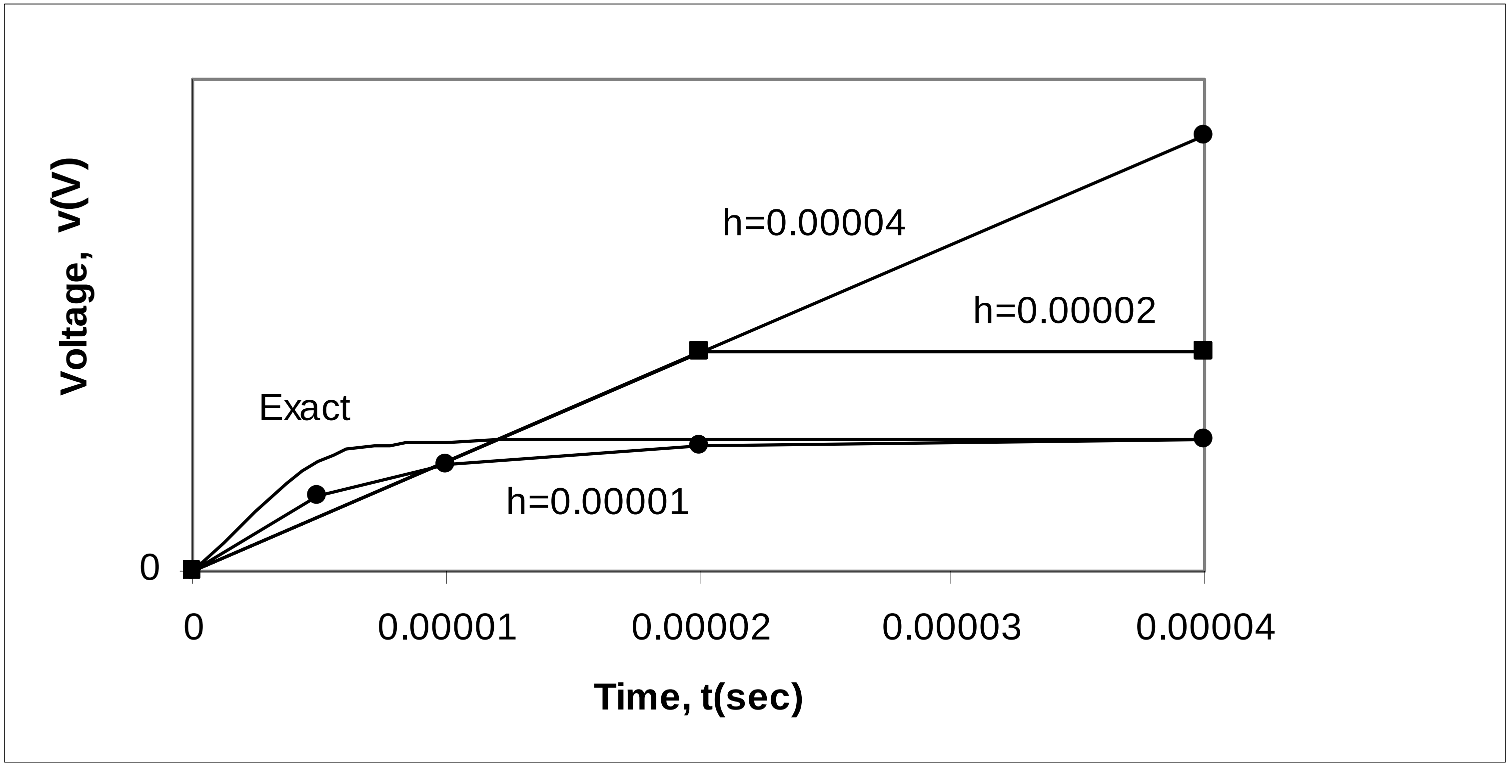

Figure 2. Heun's method results for different step sizes

Effect of step size

Table 1. Effect of step size for Heun's method

Step h Size	$v(0.00004)$	E_t	$\lvert\in_t\rvert$ %
0.00004	53.307	-37.333	233.71
0.00002	26.400	-10.426	65.269
0.00001	15.979	-0.0050000	0.031301
0.000005	15.917	0.057000	0.35683
0.0000025	15.968	0.0060000	0.037561

$v(0.00004) = 15.974V$ (exact)

Effects of step size on Heun's Method

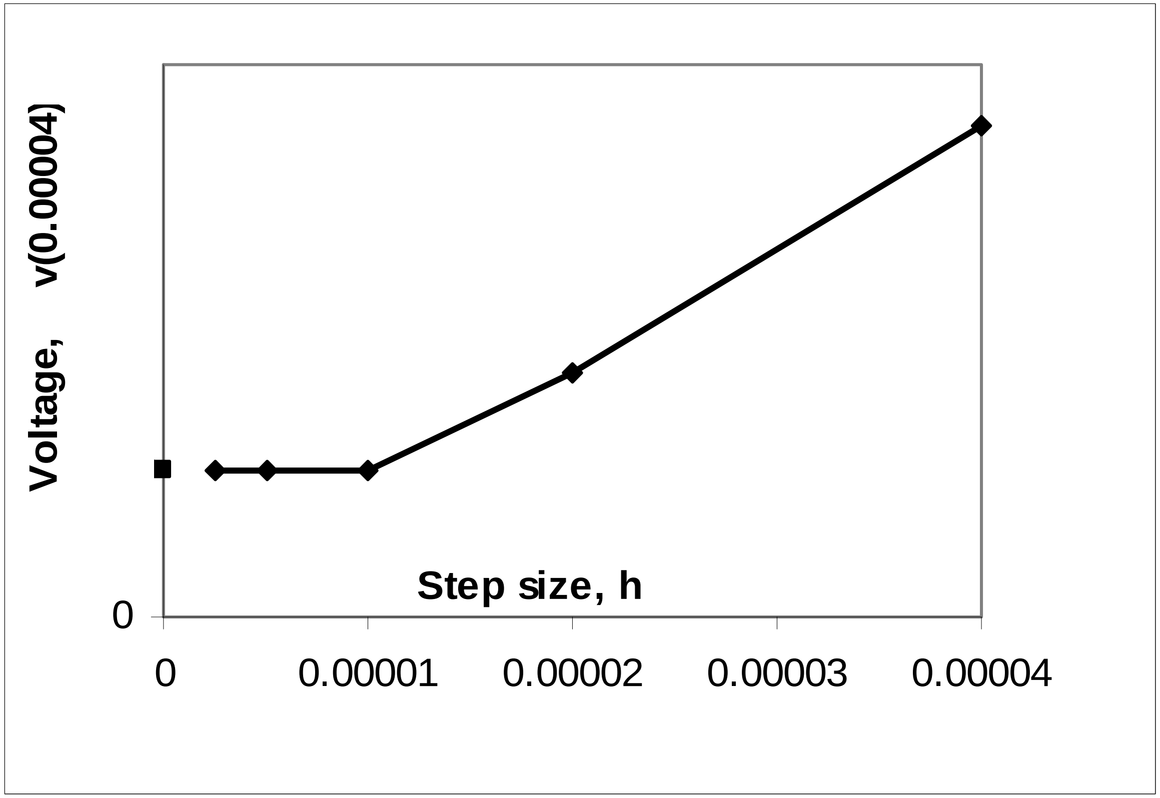

Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$v(0.00004)$			
	Euler	Heun	Midpoint	Ralston
0.00004	106.64	53.307	-0.026667	35.529
0.00002	53.307	26.400	-0.026667	17.751
0.00001	26.640	15.979	11.642	15.363
0.000005	15.995	15.917	15.917	15.917
0.0000025	15.992	15.968	15.968	15.968

$$v(0.00004) = 15.974V$$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\lvert\in_t\rvert\%$			
	Euler	Heun	Midpoint	Ralston
0.00004	567.59	233.71	100.17	122.47
0.00002	233.71	65.269	100.17	11.152
0.00001	66.771	0.031301	27.101	3.8009
0.000005	0.13146	0.35683	0.33187	0.33187
0.0000025	0.11268	0.037561	0.012523	0.012523

$$v(0.00004) = 15.974V$$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

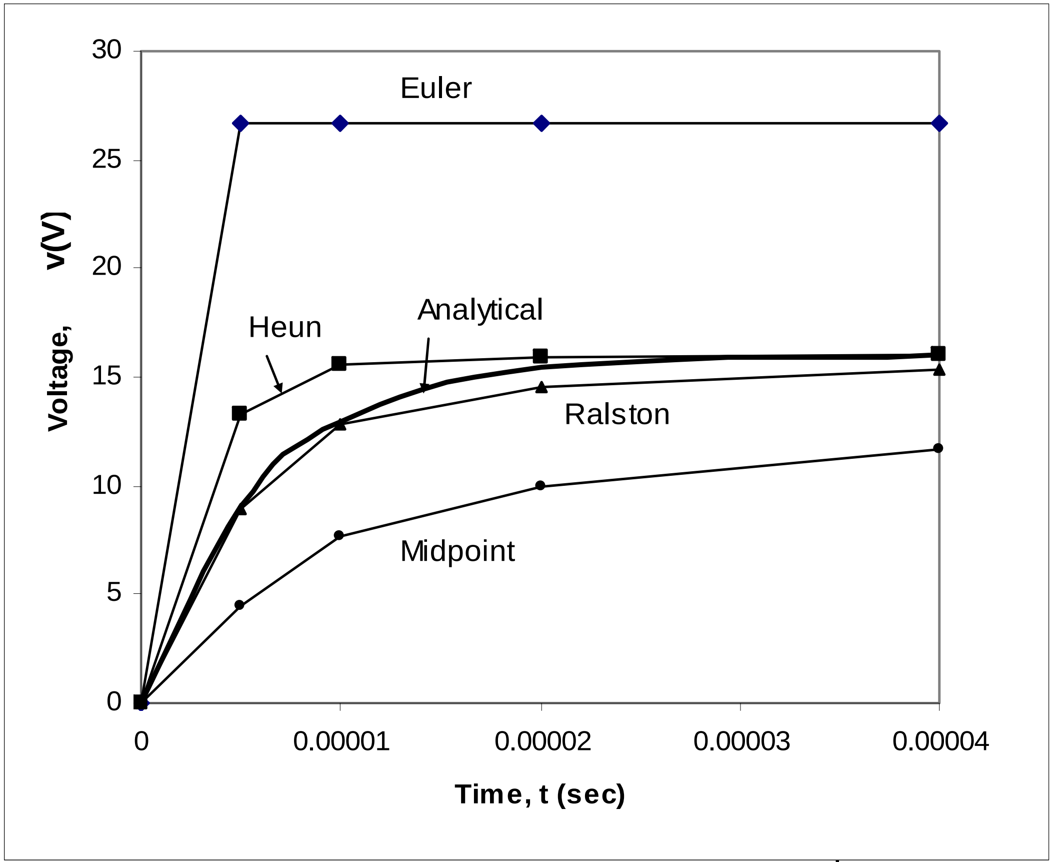

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.