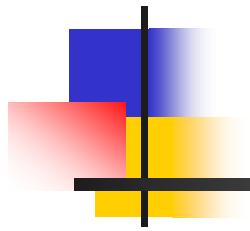


Ordinary Differential Equations



Topic: Runge-Kutta 4th Order
Method

Major: Electrical Engineering

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Runge-Kutta 4th Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h\right)$$

$$k_4 = f(x_i + h, y_i + k_3h)$$



How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$



Example

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of $150 \mu\text{F}$, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt} = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left(\frac{|18 \cos(120\pi(t))| - 2 - v(t)}{0.04}, 0 \right) \right\} \quad v(0) = 0$$

Find voltage across the capacitor at $t = 0.00004\text{s}$. Use step size $h = 0.00002$

$$\frac{dv}{dt} = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left(\frac{|18 \cos(120\pi(t))| - 2 - v}{0.04}, 0 \right) \right\}$$

$$f(t, v) = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left(\frac{|18 \cos(120\pi(t))| - 2 - v}{0.04}, 0 \right) \right\}$$

$$v_{i+1} = v_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)h$$



Solution

Step 1: $i = 0$ $t_0 = 0$ $v_0 = 0V$

$$k_1 = f(t_0, v_0) = f(0, 0) = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left(\frac{|18 \cos(120\pi(0))| - 2 - (0)}{0.04}, 0 \right) \right\} = 2.6667 \times 10^6$$

$$k_2 = f \left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_1h \right) = f \left(0 + \frac{1}{2}(0.00002), 0 + \frac{1}{2}(2.6660 \times 10^6)0.00002 \right) = f(0.00001, 26.660)$$
$$= \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left(\frac{|18 \cos(120\pi(0.00001))| - 2 - (26.660)}{0.04}, 0 \right) \right\} = -666.67$$

$$k_3 = f \left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_2h \right) = f \left(0 + \frac{1}{2}(0.00002), 0 + \frac{1}{2}(-666.67)0.00002 \right) = f(0.00001, -0.0066667)$$
$$= \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left(\frac{|18 \cos(120\pi(0.00001))| - 2 - (-0.0066667)}{0.04}, 0 \right) \right\} = 2.6671 \times 10^6$$

$$k_4 = f \left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_3h \right) = f \left(0 + \frac{1}{2}0.00002, 0 + \frac{1}{2}(2.6671 \times 10^6)0.00002 \right) = f(0.00001, 26.671)$$
$$= \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max \left(\frac{|18 \cos(120\pi(0.00001))| - 2 - (26.671)}{0.04}, 0 \right) \right\} = -666.67$$



Solution Cont

$$\begin{aligned}v_1 &= v_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\&= 0 + \frac{1}{6}(2.6660 \times 10^6 + 2(-666.67) + 2(2.6671 \times 10^6) + (-666.67))0.00002 \\&= 0 + \frac{1}{6}(7.9922 \times 10^6)0.00002 \\&= 26.641V\end{aligned}$$

v_1 is the approximate voltage at

$$t = t_1 = t_0 + h = 0 + 0.00002 = 0.00002$$

$$v_1 = v(0.00002) \approx 26.641V$$



Solution Cont

Step 2: $i = 1, t_1 = 0.00002, v_1 = 26.641V$

$$k_1 = f(t_1, v_1) = f(0.00002, 26.641) = \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max\left(\frac{|18 \cos(120\pi(0.00002))| - 2 - (26.641)}{0.04}, 0 \right) \right\} = -666.67$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_1h\right) = f\left(0.00002 + \frac{1}{2}(0.00002), 26.641 + \frac{1}{2}(-666.67)0.00002\right) = f(0.00003, 26.634)$$
$$= \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max\left(\frac{|18 \cos(120\pi(0.00003))| - 2 - (26.634)}{0.04}, 0 \right) \right\} = -666.67$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_2h\right) = f\left(0.00002 + \frac{1}{2}(0.00002), 26.641 + \frac{1}{2}(-666.67)0.00002\right) = f(0.00003, 26.634)$$
$$= \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max\left(\frac{|18 \cos(120\pi(0.00003))| - 2 - (26.634)}{0.04}, 0 \right) \right\} = -666.67$$

$$k_4 = f(t_1 + h, v_1 + k_3h) = f\left(0.00002 + \frac{1}{2}(0.00002), 26.641 + \frac{1}{2}(-666.67)0.00002\right) = f(0.00003, 26.634)$$
$$= \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max\left(\frac{|18 \cos(120\pi(0.00003))| - 2 - (26.634)}{0.04}, 0 \right) \right\} = -666.67$$



Solution Cont

$$\begin{aligned}v_2 &= v_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\&= 26.641 + \frac{1}{6}(-666.67 + 2(-666.67) + 2(-666.67) + (-666.67))0.00002 \\&= 26.641 + \frac{1}{6}(-4000.0)0.00002 \\&= 26.627V\end{aligned}$$

v_2 is the approximate voltage at

$$t_2 = t_1 + h = 0.00002 + 0.00002 = 0.00004$$

$$v_2 = v(0.00004) \approx 26.627V$$



Solution Cont

The **exact** solution to the differential equation at $t=0.00004$ seconds is

$$v(0.00004) = 15.974V$$

Comparison with exact results

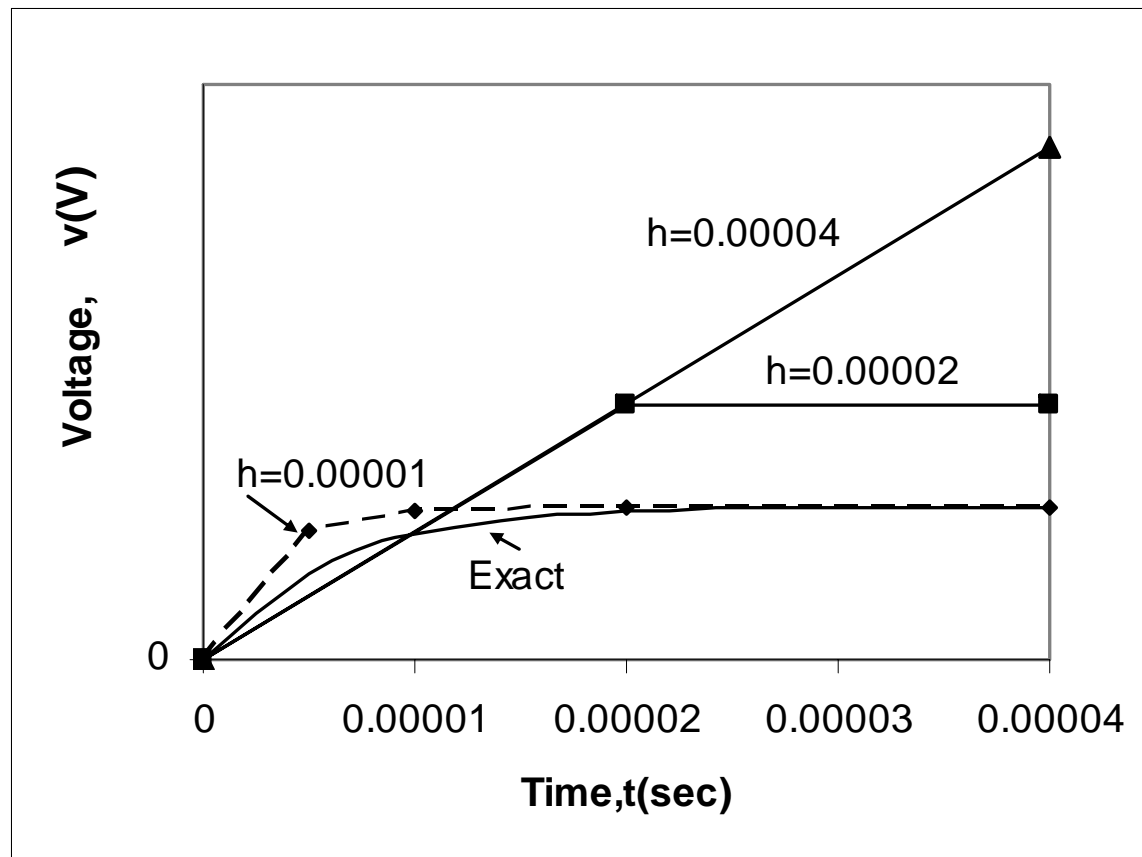


Figure 1. Comparison of Runge-Kutta 4th order method with exact solution



Effect of step size

Table 1. Value of voltage at time, $t=0.00004$ s for different step sizes

Step h	$v(0.00004)$	E_t	$ \epsilon_t \%$
0.00004	53.335	-37.361	233.89
0.00002	26.647	-10.673	66.815
0.00001	15.985	-0.011000	0.068862
0.000005	15.973	0.0010000	0.0062600
0.0000025	15.974	0	0

$$v(0.00004) = 15.974V \quad (\text{exact})$$

Effects of step size on Runge-Kutta 4th Order Method

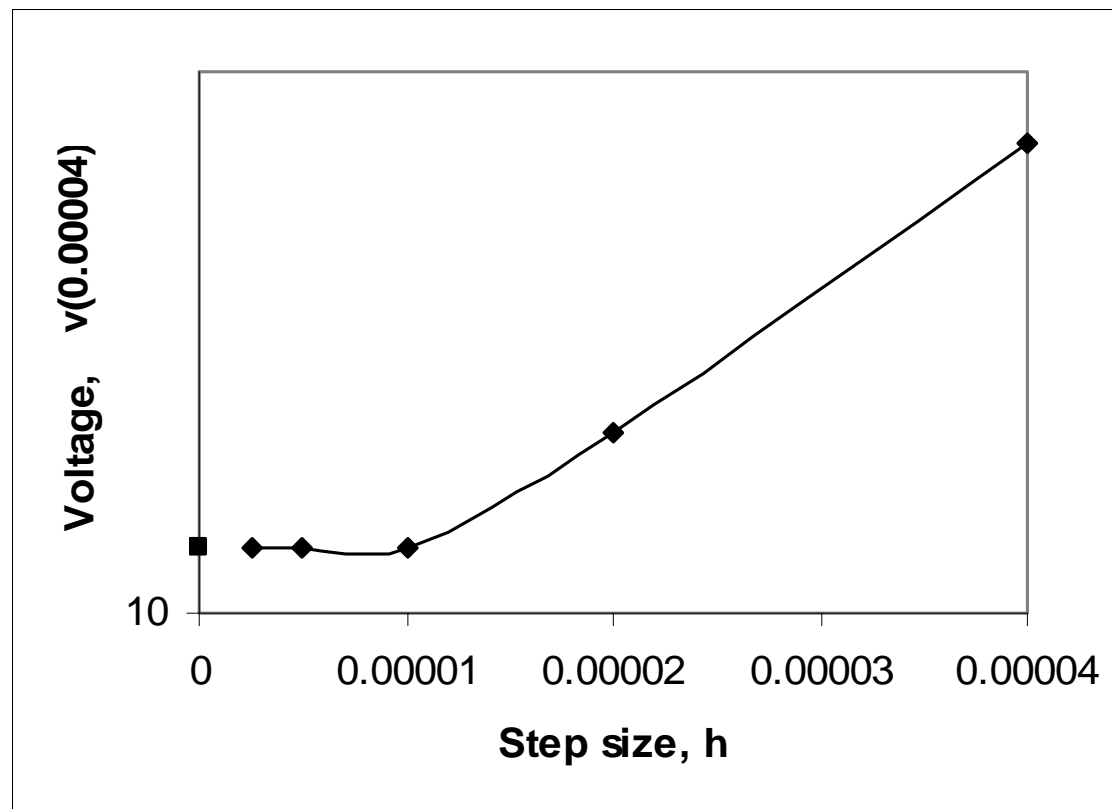


Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

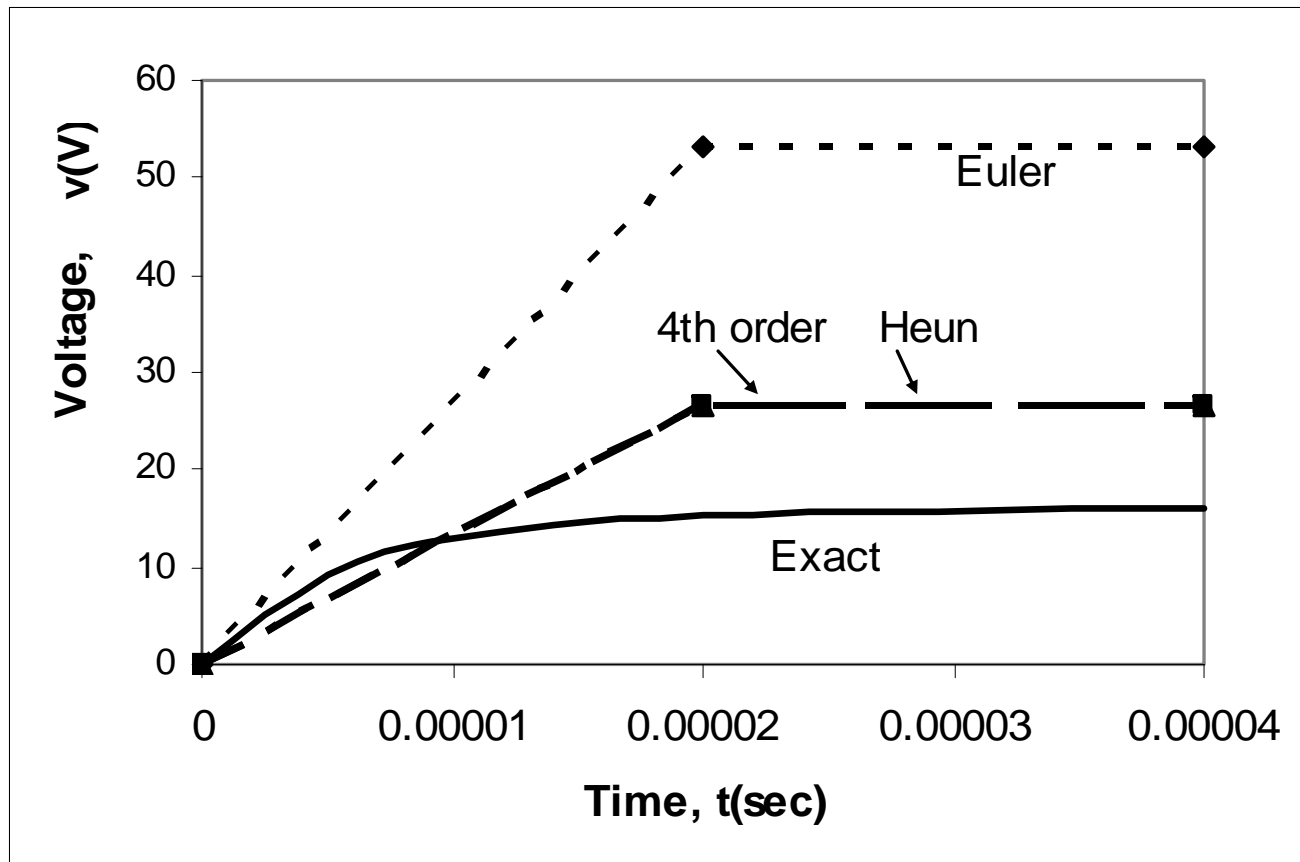


Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.