Differentiation-Discrete Functions

Major: Electrical Engineering

Authors: Autar Kaw, Sri Harsha Garapati

Forward Difference Approximation

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For a finite 'Δx'

$$f'(x) \cong \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Graphical Representation Of Forward Difference Approximation

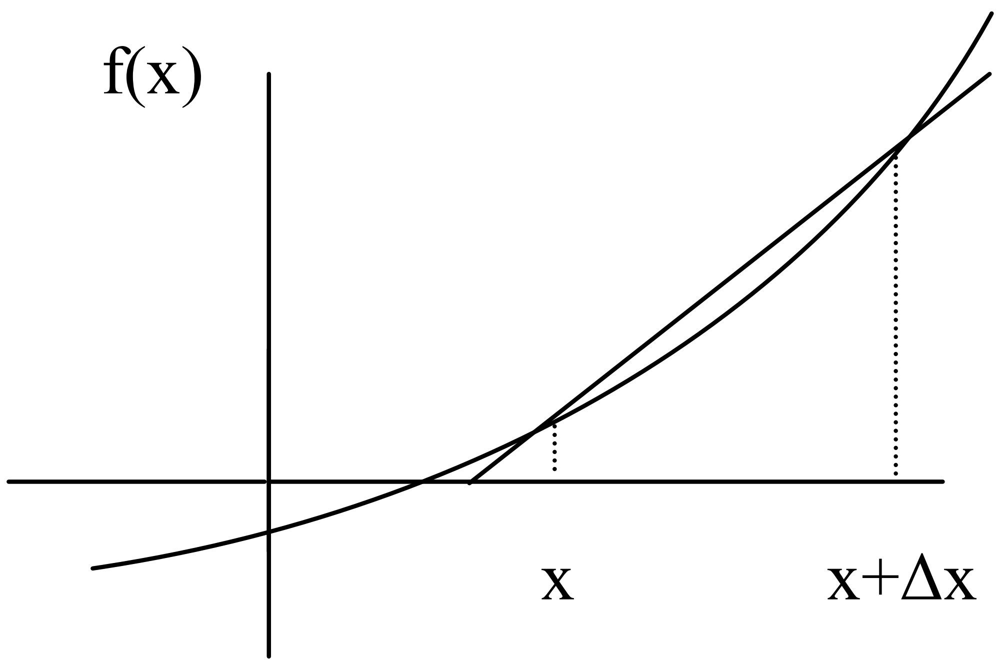

Figure 1: Graphical Representation of forward difference approximation of first derivative

Example 1

To increase the reliability and life of a switch, one needs to turn the switch off as close to zero crossing as possible. To find this time of zero crossing, the value of $E(t)/E'(t)$ is to be found at all times given in the table below, where $E(t)$ is the voltage and t is the time. To keep the problem simple, you are asked to find the approximate value of $E(t)/E'(t)$ at t=10.

Use Forward Divided Difference approximation of the first derivative to calculate $E(t)/E'(t)$ at t=10. Use a step size of $\Delta t = 1$.

Example 1 Cont.

Time	Voltage	Time	Voltage
1	0.62161	13	-0.210796
2	0.362358	14	0.087499
3	0.070737	15	0.377978
4	-0.227202	16	0.634693
5	-0.504846	17	0.834713
6	-0.737394	18	0.96017
7	-0.904072	19	0.999859
8	-0.989992	20	0.950233
9	-0.98748	21	0.815725
10	-0.896758	22	0.608351
11	-0.725932	23	0.346635
12	-0.490261	24	0.053955

Example 1 Cont.

$$E'(t_i) \cong \frac{E(t_{i+1}) - E(t_i)}{\Delta t}$$

$$t_i = 10$$

$$E'(10) = \frac{E(11) - E(10)}{1}$$

$$= \frac{-0.725932 - (-0.896758)}{1}$$

$$= 0.170826 V$$

$$\frac{E(10)}{E'(10)} = \frac{-0.896758}{0.170826}$$

$$= -5.2495 \sec$$

Direct Fit Polynomials

In this method, given $'n+1'$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1 x + \ldots\ldots + a_{n-1} x^{n-1} + a_n x^n$$

To find the first derivative,

$$P_n'(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \ldots\ldots + (n-1)a_{n-1} x^{n-2} + n a_n x^{n-1}$$

Similarly other derivatives can be found.

Example 2-Direct Fit Polynomials

To increase the reliability and life of a switch, one needs to turn the switch off as close to zero crossing as possible. To find this time of zero crossing, the value of $E(t)/E'(t)$ is to be found at all times given in the table below, where $E(t)$ is the voltage and t is the time. To keep the problem simple, you are asked to find the approximate value of $E(t)/E'(t)$ at t=10.

Using the third order polynomial interpolant for Voltage, find the value of $E(t)/E'(t)$ at t=10.

Example 2-Direct Fit Polynomials cont.

Time	Voltage	Time	Voltage
1	0.62161	13	-0.210796
2	0.362358	14	0.087499
3	0.070737	15	0.377978
4	-0.227202	16	0.634693
5	-0.504846	17	0.834713
6	-0.737394	18	0.96017
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8	-0.989992	20	0.950233
9	-0.98748	21	0.815725
10	-0.896758	22	0.608351
11	-0.725932	23	0.346635
12	-0.490261	24	0.053955

Example 2-Direct Fit Polynomials cont.

For the third order polynomial (also called cubic interpolation), we choose the velocity given by

$$E(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Since we want to find the voltage at t=10, and we are using a third order polynomial, we need to choose the four points closest to $t = 10$ and that also bracket $t = 10$ to evaluate it.

The four points are t0=8, t1=9, t2=10 and t3=11.

$$t_o = 8, \ E(t_o) = -0.989992$$

$$t_1 = 9, \ E(t_1) = -0.98748$$

$$t_2 = 10, \ E(t_2) = -0.896758$$

$$t_3 = 11, \ E(t_3) = -0.725932$$

Example 2-Direct Fit Polynomials cont.

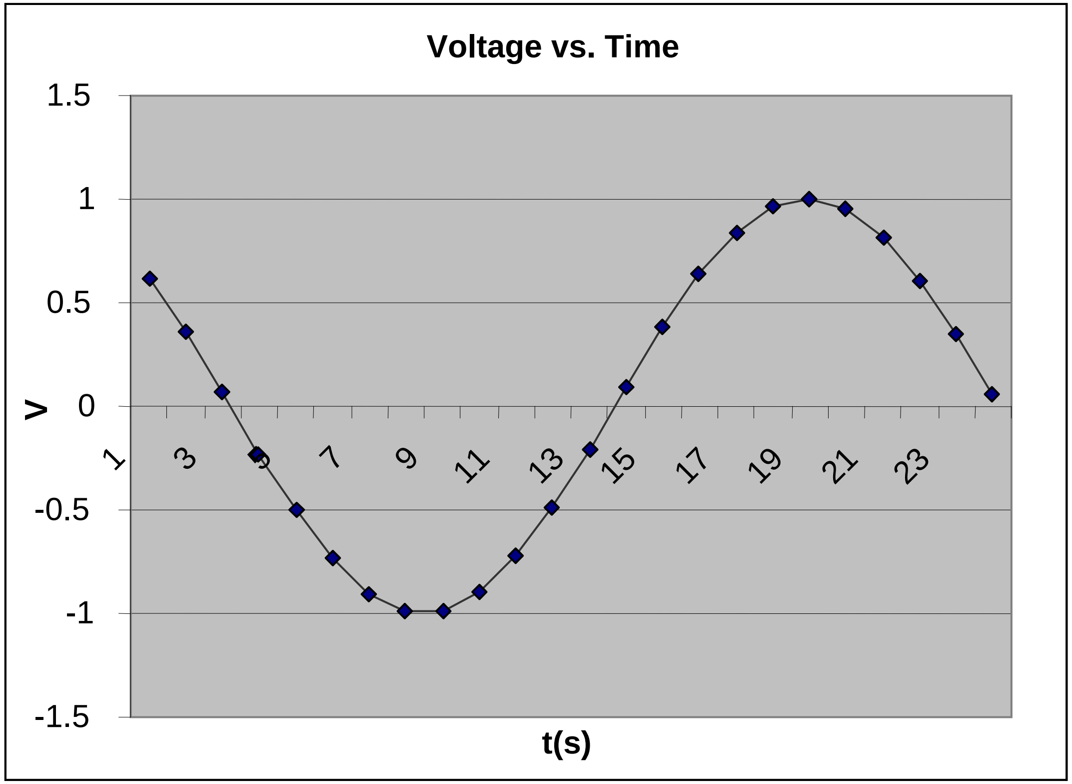

Figure 1: Graph of Voltage of the switch vs. time

Example 2-Direct Fit Polynomials cont.

such that

$$E(8) = -0.989992 = a_0 + a_1(8) + a_2(8)^2 + a_3(8)^3$$

$$E(9) = -0.98748 = a_0 + a_1(9) + a_2(9)^2 + a_3(9)^3$$

$$E(10) = -0.896758 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$

$$E(11) = -0.725932 = a_0 + a_1(11) + a_2(11)^2 + a_3(11)^3$$

Writing the four equations in matrix form, we have

Example 2-Direct Fit Polynomials cont.

$$\begin{bmatrix} 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -0.989992 \\ -0.98748 \\ -0.896758 \\ -0.725932 \end{bmatrix}$$

Solving the above four equations gives

$$a_0 = 3.1382$$

$$a_1 = \text{-}1.0742$$

$$a_2 = 0.080582$$

$$a_3 = \text{-}0.0013510$$

Example 2-Direct Fit Polynomials cont.

Hence

$$E(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$= 3.1382 - 1.0742t + 0.080582t^2 - 0.0013510t^3, \quad 8 \le t \le 11$$

The derivative of voltage at t=10 is given by

$$E'(10) = \left.\frac{d}{dt}E(t)\right|_{t=10}$$

Given that

$$E(t) = 3.1382 - 1.0742\,t + 0.080582\,t^2 - 0.0013510\,t^3, 10 \le t \le 22.5$$

$$E'(t) = \frac{d}{dt}E(t)$$

Example 2-Direct Fit Polynomials cont.

$$= \frac{d}{dt}\left(3.13812 - 1.0742t + 0.080582t^2 - 0.0013510t^3\right)$$

$$= -1.0742 + 0.16116t - 0.0040530t^2, \quad 8 \le t \le 11$$

$$E'(10) = -1.0742 + 0.16116\,t - 0.0040530\,t^2$$

$$= 0.13210 V/\sec$$

$$\frac{E(10)}{E'(10)} = \frac{-0.896758}{0.13210}$$

$$= -6.7885 \sec$$

Lagrange Polynomial

In this method, given $(x_1, y_1), \ldots, (x_n, y_n)$, one can fit a $(n-1)^{th}$ order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Lagrange Polynomial Cont.

Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ is

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating equation (2) gives

Lagrange Polynomial Cont.

$$f_2'(x) = \frac{2x - (x_1 + x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - (x_0 + x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2x - (x_0 + x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Example 3

To increase the reliability and life of a switch, one needs to turn the switch off as close to zero crossing as possible. To find this time of zero crossing, the value of $E(t)/E'(t)$ is to be found at all times given in the table below, where $E(t)$ is the voltage and t is the time. To keep the problem simple, you are asked to find the approximate value of $E(t)/E'(t)$ at t=10.

Use second order Lagrangian polynomial interpolation to calculate $E(t)/E'(t)$ at t=10.

Example 3 Cont.

Time	Voltage	Time	Voltage
1	0.62161	13	-0.210796
2	0.362358	14	0.087499
3	0.070737	15	0.377978
4	-0.227202	16	0.634693
5	-0.504846	17	0.834713
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10	-0.896758	22	0.608351
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12	-0.490261	24	0.053955

Example 3 Cont.

Solution:

$$E(t)=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)E(t_0)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)E(t_1)+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)E(t_2)$$

$$E^{'}(t)=\frac{2t-(t_1+t_2)}{(t_0-t_1)(t_0-t_2)}E(t_0)\ +\frac{2t-(t_0+t_2)}{(t_1-t_0)(t_1-t_2)}E(t_1)\ +\frac{2t-(t_0+t_1)}{(t_2-t_0)(t_2-t_1)}E(t_2)$$

$$E^{'}(10)=\frac{2(10)-(11+12)}{(9-11)(9-12)}(-0.98748)+\frac{2(10)-(9+12)}{(11-9)(11-12)}(-0.725932)\ +\frac{2(10)-(9+11)}{(12-9)(12-11)}(-0.490261)$$

$$=-0.5(-0.98748)+0.5(-0.72593)+0(-0.49026)$$

Example 3 Cont.

$$= 0.13077 V / \sec$$

$$\frac{E(10)}{E^{'}(10)} = \frac{-0.896758}{0.13077}$$

$$= -6.8573 \ \sec$$