

## Chapter 02.03

# Differentiation of Discrete Functions

*After reading this chapter, you should be able to:*

- 1. find approximate values of the first derivative of functions that are given at discrete data points, and*
- 2. use Lagrange polynomial interpolation to find derivatives of discrete functions.*

To find the derivatives of functions that are given at discrete points, several methods are available. Although these methods are mainly used when the data is spaced unequally, they can be used for data that is spaced equally as well.

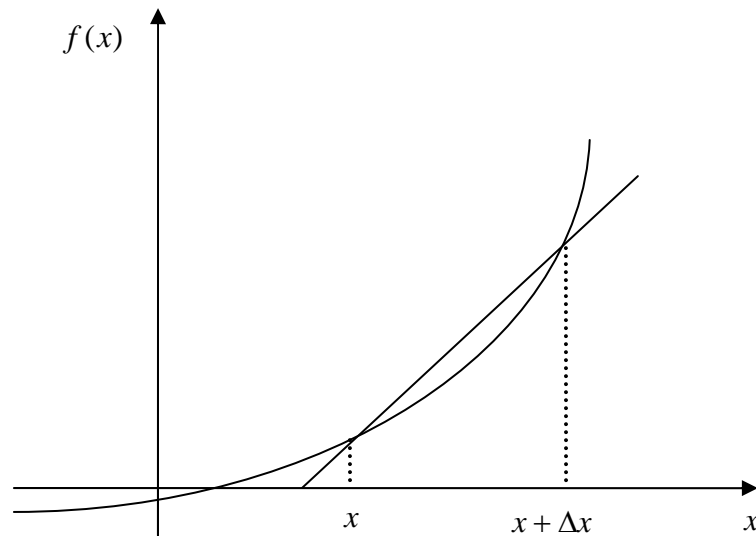
### **Forward Difference Approximation of the First Derivative**

We know

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite  $\Delta x$ ,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



**Figure 1** Graphical representation of forward difference approximation of first derivative.

So given  $n+1$  data points  $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ , the value of  $f'(x)$  for  $x_i \leq x \leq x_{i+1}$ ,  $i = 0, \dots, n-1$ , is given by

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

### Example 1

To increase the reliability and life of a switch, one needs to turn the switch off as close to the zero-crossing as possible. To find this time of zero-crossing, the value of  $E(t)/E'(t)$  is to be found at all times given in Table 1, where  $E(t)$  is the voltage and  $t$  is the time. To keep the problem simple, you are asked to find the approximate value of  $E(t)/E'(t)$  at  $t = 10$ .

**Table 1** Voltage as a function of time.

Time, $t$ (s)	Voltage, $E(t)$ (V)	Time, $t$ (s)	Voltage, $E(t)$ (V)
1	0.62161	13	-0.210796
2	0.362358	14	0.087499
3	0.070737	15	0.377978
4	-0.227202	16	0.634693
5	-0.504846	17	0.834713
6	-0.737394	18	0.96017
7	-0.904072	19	0.999859
8	-0.989992	20	0.950233
9	-0.98748	21	0.815725
10	-0.896758	22	0.608351
11	-0.725932	23	0.346635
12	-0.490261	24	0.053955

Use the forward divided difference approximation of the first derivative to calculate  $E(t)/E'(t)$  at  $t = 10$ . Use a step size of  $\Delta t = 1$ .

### Solution

$$\begin{aligned}
 E'(t_i) &\approx \frac{E(t_{i+1}) - E(t_i)}{\Delta t} \\
 t_i &= 10 \\
 t_{i+1} &= 11 \\
 \Delta t &= t_{i+1} - t_i \\
 &= 11 - 10 \\
 &= 1 \\
 E'(10) &\approx \frac{E(11) - E(10)}{1} \\
 &= \frac{-0.725932 - (-0.896758)}{1} \\
 &= 0.170826 \text{ V/s} \\
 \frac{E(10)}{E'(10)} &= \frac{-0.896758}{0.170826} \\
 &= -5.2495 \text{ s}
 \end{aligned}$$

### Direct Fit Polynomials

In this method, given  $n + 1$  data points  $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ , one can fit a  $n^{\text{th}}$  order polynomial given by

$$P_n(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n$$

To find the first derivative,

$$P'_n(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2x + \dots + (n-1)a_{n-1}x^{n-2} + na_nx^{n-1}$$

Similarly, other derivatives can also be found.

### Example 2

To increase the reliability and life of a switch, one needs to turn the switch off as close to the zero-crossing as possible. To find this time of zero-crossing, the value of  $E(t)/E'(t)$  is to be found at all times given in Table 2, where  $E(t)$  is the voltage and  $t$  is the time. To keep the problem simple, you are asked to find the approximate value of  $E(t)/E'(t)$  at  $t = 10$ .

**Table 2** Voltage as a function of time.

Time, $t$ (s)	Voltage, $E(t)$ (V)	Time, $t$ (s)	Voltage, $E(t)$ (V)
1	0.62161	13	-0.210796
2	0.362358	14	0.087499
3	0.070737	15	0.377978
4	-0.227202	16	0.634693
5	-0.504846	17	0.834713
6	-0.737394	18	0.96017
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Using a third order polynomial interpolant for Voltage, find the value of  $E(t)/E'(t)$  at  $t = 10$ .

**Solution**

For a third order polynomial interpolation (also called cubic interpolation), we choose the voltage given by

$$E(t) = a_0 + a_1t + a_2t^2 + a_3t^3$$

Since we want to find the voltage at  $t = 10$ , and we are using a third order polynomial, we need to choose the four points closest to  $t = 10$  that also bracket  $t = 10$  to evaluate it.

The four points are  $t_0 = 8$ ,  $t_1 = 9$ ,  $t_2 = 10$  and  $t_3 = 11$ .

$$t_0 = 8, E(t_0) = -0.989992$$

$$t_1 = 9, E(t_1) = -0.98748$$

$$t_2 = 10, E(t_2) = -0.896758$$

$$t_3 = 11, E(t_3) = -0.725932$$

such that

$$E(8) = -0.989992 = a_0 + a_1(8) + a_2(8)^2 + a_3(8)^3$$

$$E(9) = -0.98748 = a_0 + a_1(9) + a_2(9)^2 + a_3(9)^3$$

$$E(10) = -0.896758 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$

$$E(11) = -0.725932 = a_0 + a_1(11) + a_2(11)^2 + a_3(11)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -0.989992 \\ -0.98748 \\ -0.896758 \\ -0.725932 \end{bmatrix}$$

Solving the above gives

$$a_0 = 3.1382$$

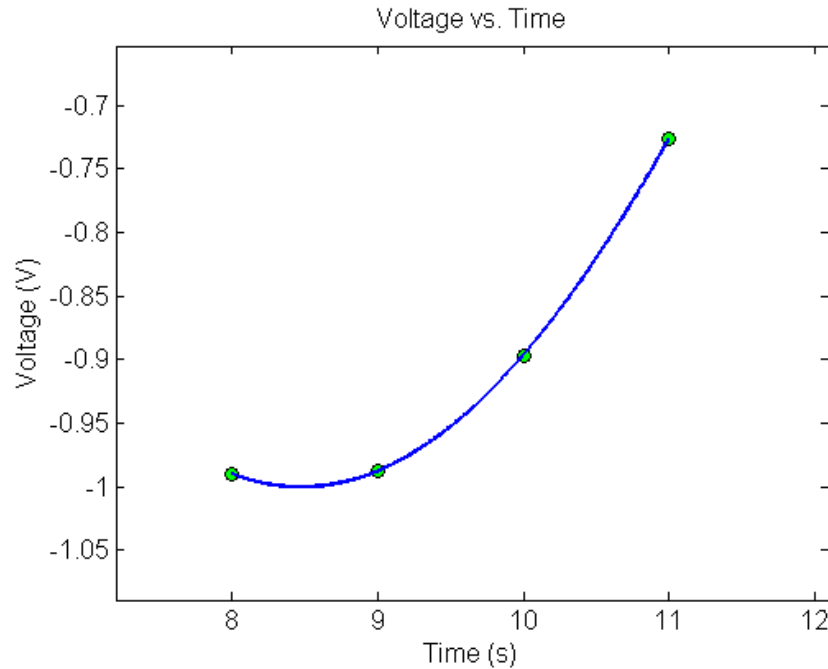
$$a_1 = -1.0742$$

$$a_2 = 0.080582$$

$$a_3 = -0.001351$$

Hence

$$\begin{aligned} E(t) &= a_0 + a_1t + a_2t^2 + a_3t^3 \\ &= 3.1382 - 1.0742t + 0.080582t^2 - 0.0013510t^3, \quad 8 \leq t \leq 11 \end{aligned}$$



**Figure 2** Graph of voltage of the switch vs. time.

The derivative of voltage at  $t = 10$  is given by

$$E'(10) = \left. \frac{d}{dt} E(t) \right|_{t=10}$$

Given that  $E(t) = 3.1382 - 1.0742t + 0.080582t^2 - 0.0013510t^3$ ,  $8 \leq t \leq 11$ ,

$$\begin{aligned} E'(t) &= \frac{d}{dt} E(t) \\ &= \frac{d}{dt} (3.1382 - 1.0742t + 0.080582t^2 - 0.0013510t^3) \\ &= -1.0742 + 0.16116t - 0.004053t^2, \quad 8 \leq t \leq 11 \end{aligned}$$

$$\begin{aligned} E'(10) &= -1.0742 + 0.16116(10) - 0.004053(10)^2 \\ &= 0.13210 \text{ V/s} \end{aligned}$$

$$\begin{aligned} \frac{E(10)}{E'(10)} &= \frac{-0.896758}{0.13210} \\ &= -6.7872 \text{ s} \end{aligned}$$

### Lagrange Polynomial

In this method, given  $(x_0, y_0), \dots, (x_n, y_n)$ , one can fit a  $n^{\text{th}}$  order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where  $n$  in  $f_n(x)$  stands for the  $n^{\text{th}}$  order polynomial that approximates the function  $y = f(x)$  and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$  is a weighting function that includes a product of  $n-1$  terms with terms of  $j=i$  omitted.

Then to find the first derivative, one can differentiate  $f_n(x)$  once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through  $(x_0, y_0), (x_1, y_1)$ , and  $(x_2, y_2)$  is

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating the above equation gives

$$f_2'(x) = \frac{2x - (x_1 + x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - (x_0 + x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2x - (x_0 + x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

### Example 3

To increase the reliability and life of a switch, one needs to turn the switch off as close to the zero-crossing as possible. To find this time of zero-crossing, the value of  $E(t)/E'(t)$  is to be found at all times given in Table 3, where  $E(t)$  is the voltage and  $t$  is the time. To keep the problem simple, you are asked to find the approximate value of  $E(t)/E'(t)$  at  $t = 10$ .

**Table 3** Voltage as a function of time.

Time, $t$ (s)	Voltage, $E(t)$ (V)	Time, $t$ (s)	Voltage, $E(t)$ (V)
1	0.62161	13	-0.210796
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12	-0.490261	24	0.053955

Use second order Lagrangian polynomial interpolation to calculate  $E(t)/E'(t)$  at  $t = 10$ .

**Solution**

For second order Lagrangian polynomial interpolation, we choose the voltage given by

$$E(t) = \left( \frac{t-t_1}{t_0-t_1} \right) \left( \frac{t-t_2}{t_0-t_2} \right) E(t_0) + \left( \frac{t-t_0}{t_1-t_0} \right) \left( \frac{t-t_2}{t_1-t_2} \right) E(t_1) + \left( \frac{t-t_0}{t_2-t_0} \right) \left( \frac{t-t_1}{t_2-t_1} \right) E(t_2)$$

Since we want to find the voltage at  $t = 10$ , and we are using a second order Lagrangian polynomial, we need to choose the three points closest to  $t = 10$  that also bracket  $t = 10$  to evaluate it. The three points are  $t_0 = 9$ ,  $t_1 = 10$ , and  $t_2 = 11$ .

Differentiating the above equation gives

$$E'(t) = \frac{2t - (t_1 + t_2)}{(t_0 - t_1)(t_0 - t_2)} E(t_0) + \frac{2t - (t_0 + t_2)}{(t_1 - t_0)(t_1 - t_2)} E(t_1) + \frac{2t - (t_0 + t_1)}{(t_2 - t_0)(t_2 - t_1)} E(t_2)$$

Hence

$$\begin{aligned} E'(10) &= \frac{2(10) - (10 + 11)}{(9 - 10)(9 - 11)} (-0.98748) + \frac{2(10) - (9 + 11)}{(10 - 9)(10 - 11)} (-0.896758) \\ &\quad + \frac{2(10) - (9 + 10)}{(11 - 9)(11 - 10)} (-0.725932) \\ &= -0.5(-0.98748) + 0(-0.896758) + 0.5(-0.725932) \\ &= 0.13077 \text{ V/s} \\ \frac{E(10)}{E'(10)} &= \frac{-0.896758}{0.13077} \\ &= -6.8573 \text{ s} \end{aligned}$$

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**DIFFERENTIATION**

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Topic	Differentiation of Discrete Functions
Summary	These are textbook notes differentiation of discrete functions
Major	Electrical Engineering
Authors	Autar Kaw, Luke Snyder
Date	June 22, 2009
Web Site	<a href="http://numericalmethods.eng.usf.edu">Uhttp://numericalmethods.eng.usf.edu</a>

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