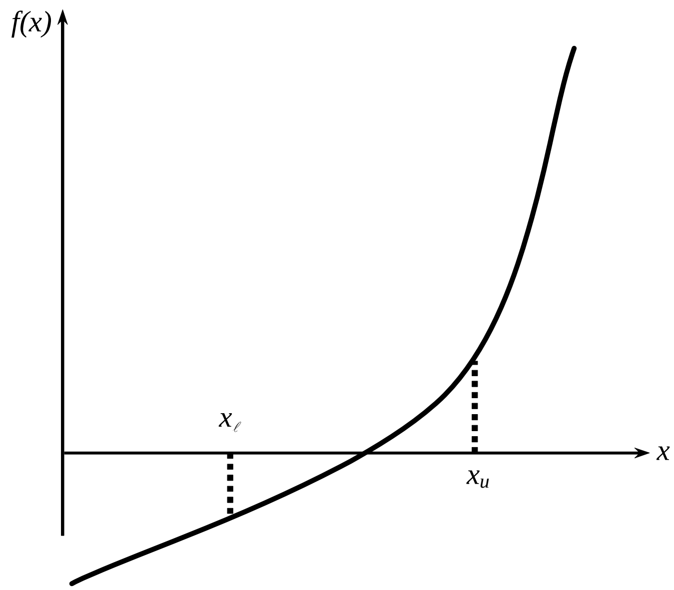
Roots of a Nonlinear Equation

Topic: Bisection Method

Major: Electrical Engineering

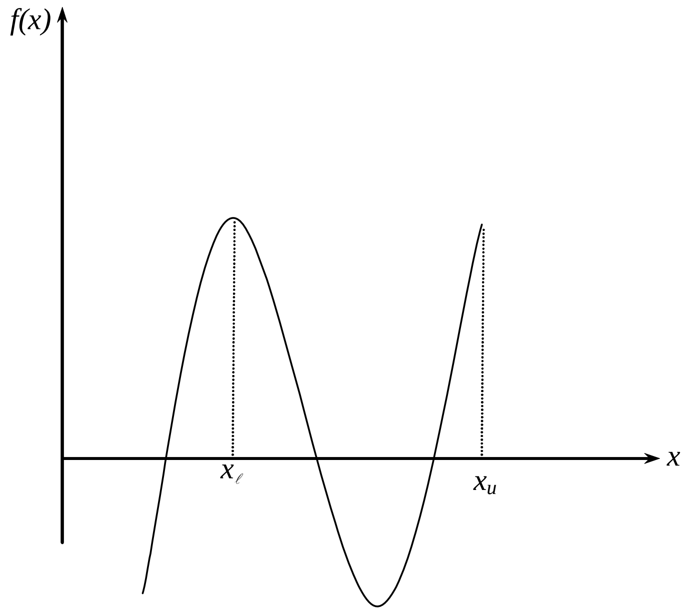
Basis of Bisection Method

Theorem: An equation $f(x)$ =0, where $f(x)$ is a real continuous function, has at least one root between x_ℓ and x_u if $f(x_\ell)\, f(x_u) < 0$.



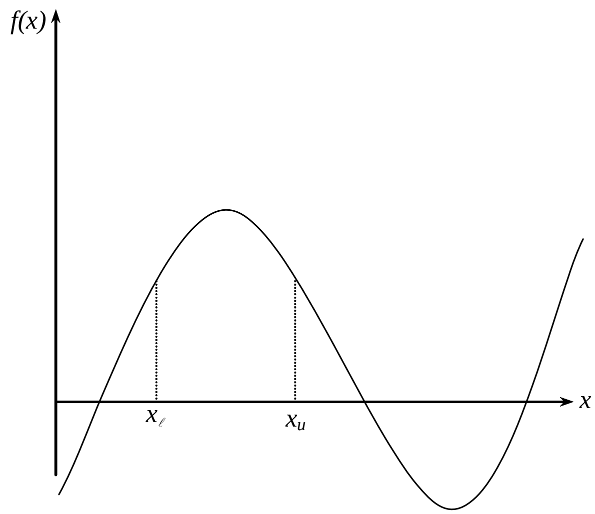
Theorem

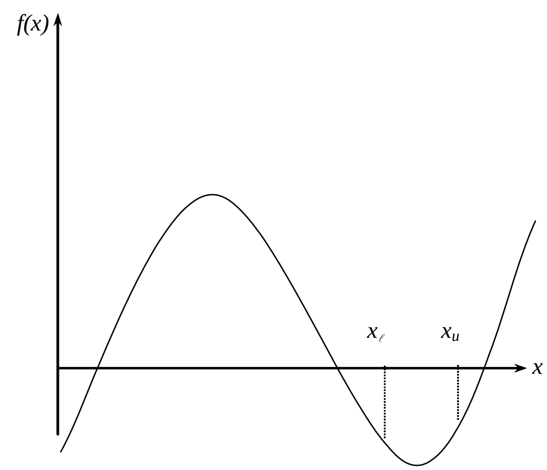
If function *f(x)* in *f(x)* =0 does not change sign between two points, roots may still exist between the two points.



Theorem

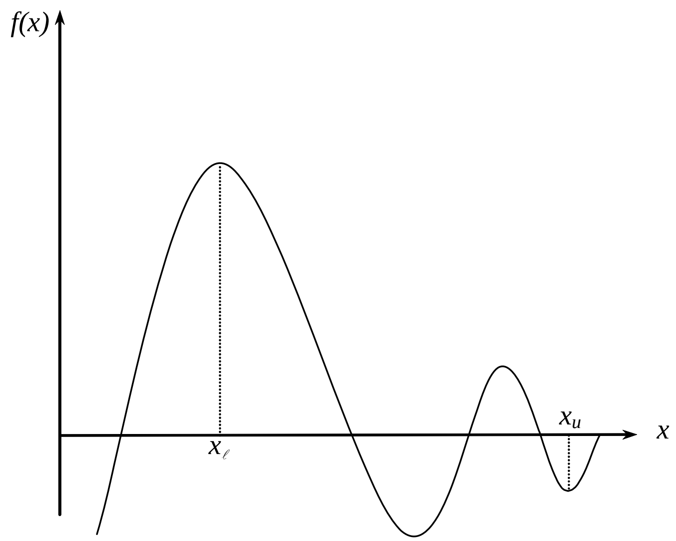
If the function $f(x)$ in $f(x)$ =0 does not change sign between two points, there may not be any roots between the two points.





Theorem

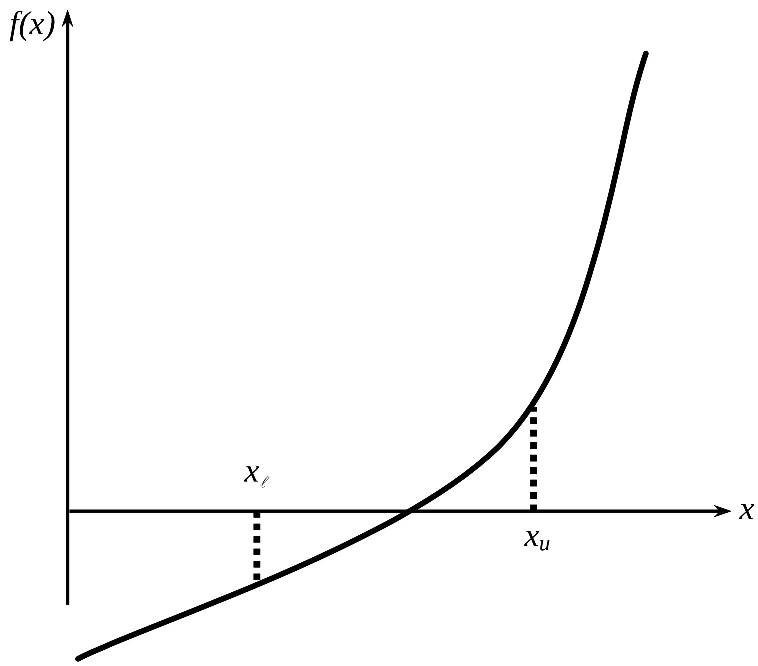
If the function $f(x)$ in $f(x) = 0$ changes sign between two points, more than one root may exist between the two points.



Algorithm for Bisection Method

Step 1

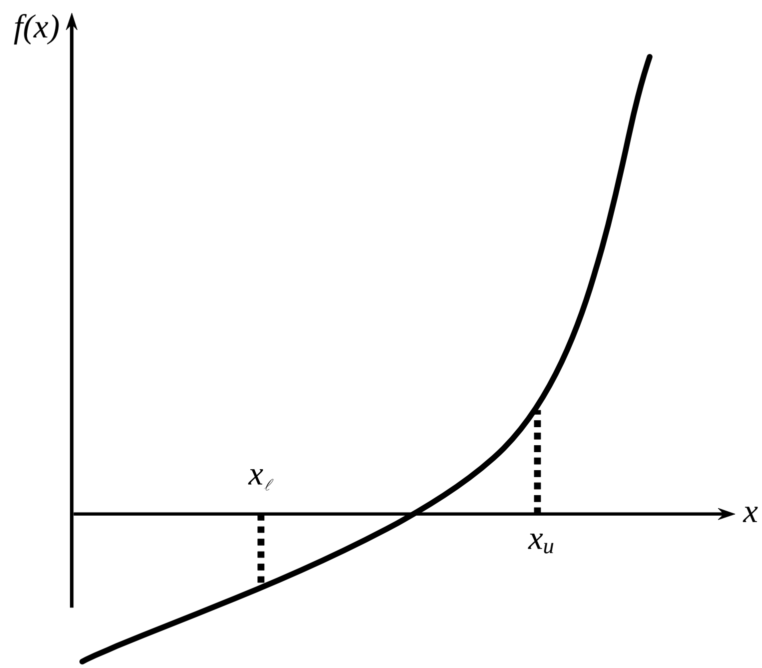
- Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)\, f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_ℓ and x_u.



Step 2

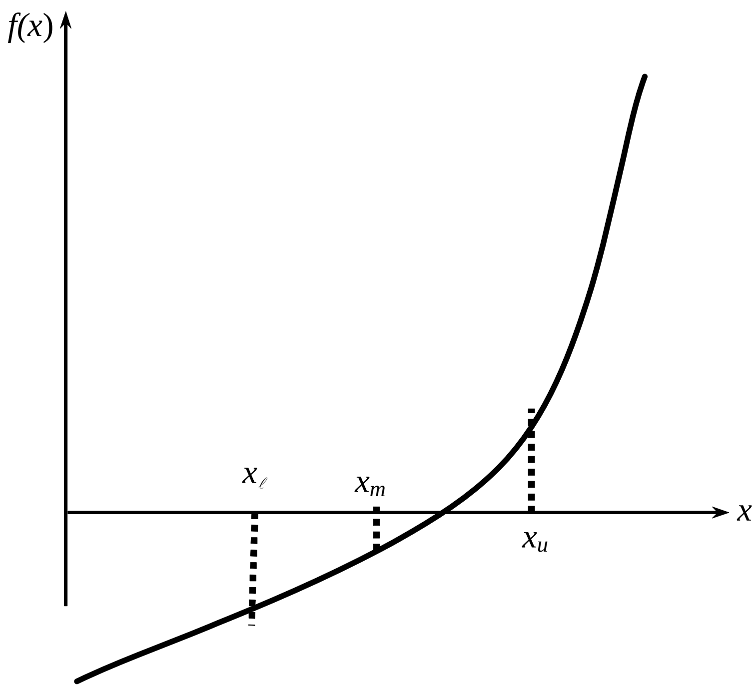
Estimate the root, x_m of the equation $f(x) = 0$ as the mid-point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$



Step 3

Now check the following



If $f(x_\ell)\, f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

If $f(x_\ell)\, f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

If $f(x_\ell)\, f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.

Step 4

New estimate

$$x_m = \frac{x_\ell + x_u}{2}$$

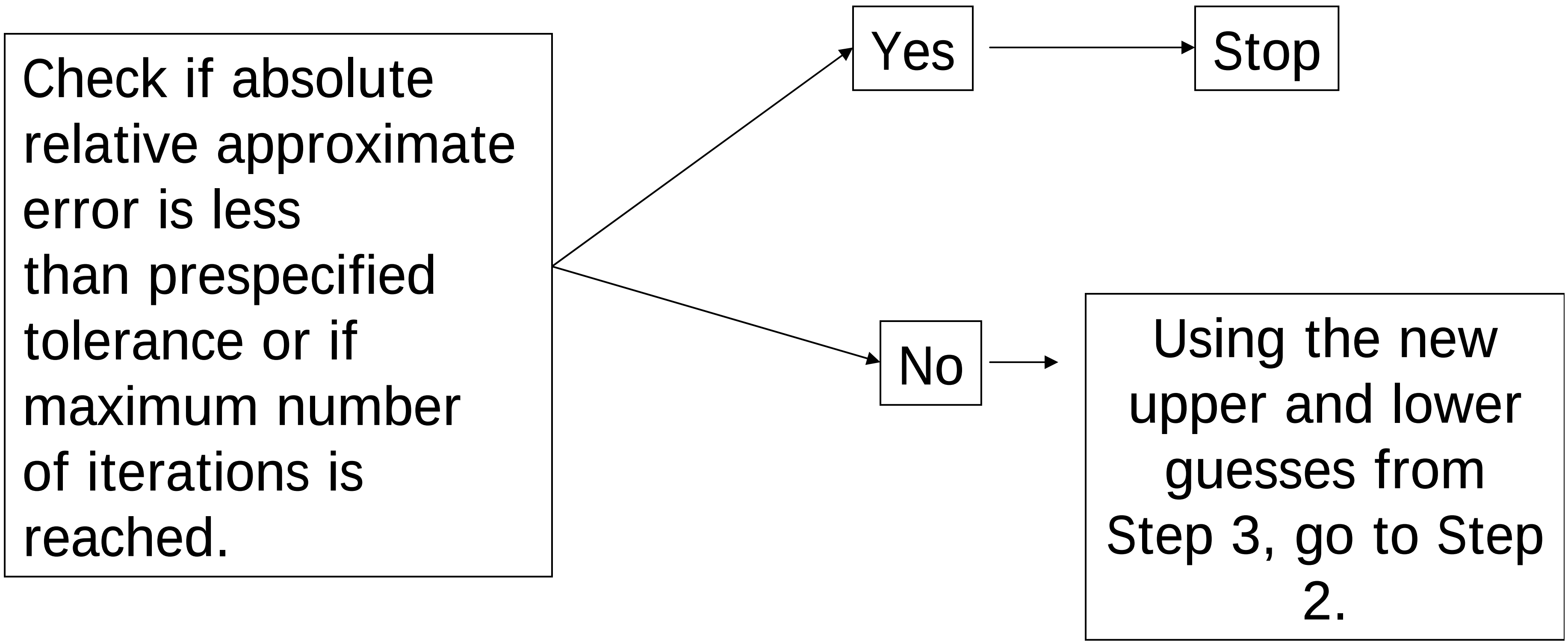
Absolute Relative Approximate Error

$$|\in_a| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$x_m^{old} = \text{previous estimate of root}$$

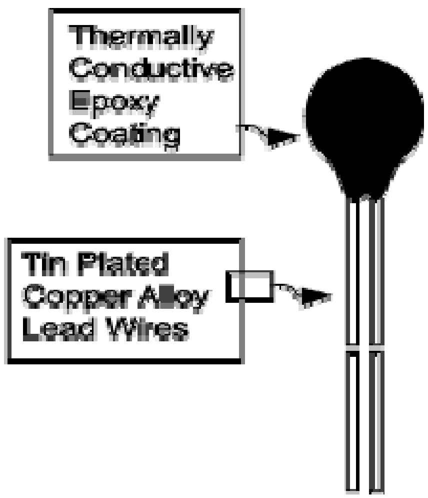
$$x_m^{new} = \text{current estimate of root}$$

Step 5


Check if absolute relative approximate error is less than prespecified tolerance or if maximum number of iterations is reached.
Yes
Stop
No
Using the new upper and lower guesses from Step 3, go to Step 2.

Example

- Thermistors are temperature-measuring devices based on the principle that the thermistor material exhibits a change in electrical resistance with a change in temperature. By measuring the resistance of the thermistor material, one can then determine the temperature.



Solution

For a 10K3A Betatherm thermistor, the relationship between the resistance '*R*' of the thermistor and the temperature is given by

$$\frac{1}{T} = 1.129241x10^{-3} + 2.341077x10^{-4}\ln(R) + 8.775468x10^{-8}\{\ln(R)\}^3$$

where note that T is in Kelvin and R is in ohms.

Solution

For the thermistor, error of no more than ±0.01^{o}C is acceptable. To find the range of the resistance that is within this acceptable limit at 19^{o}C, we need to solve

$$\frac{1}{19.01+273.15} = 1.129241x10^{-3} + 2.341077x10^{-4}\ln(R) + 8.775468x10^{-8}\{\ln(R)\}^{3}$$

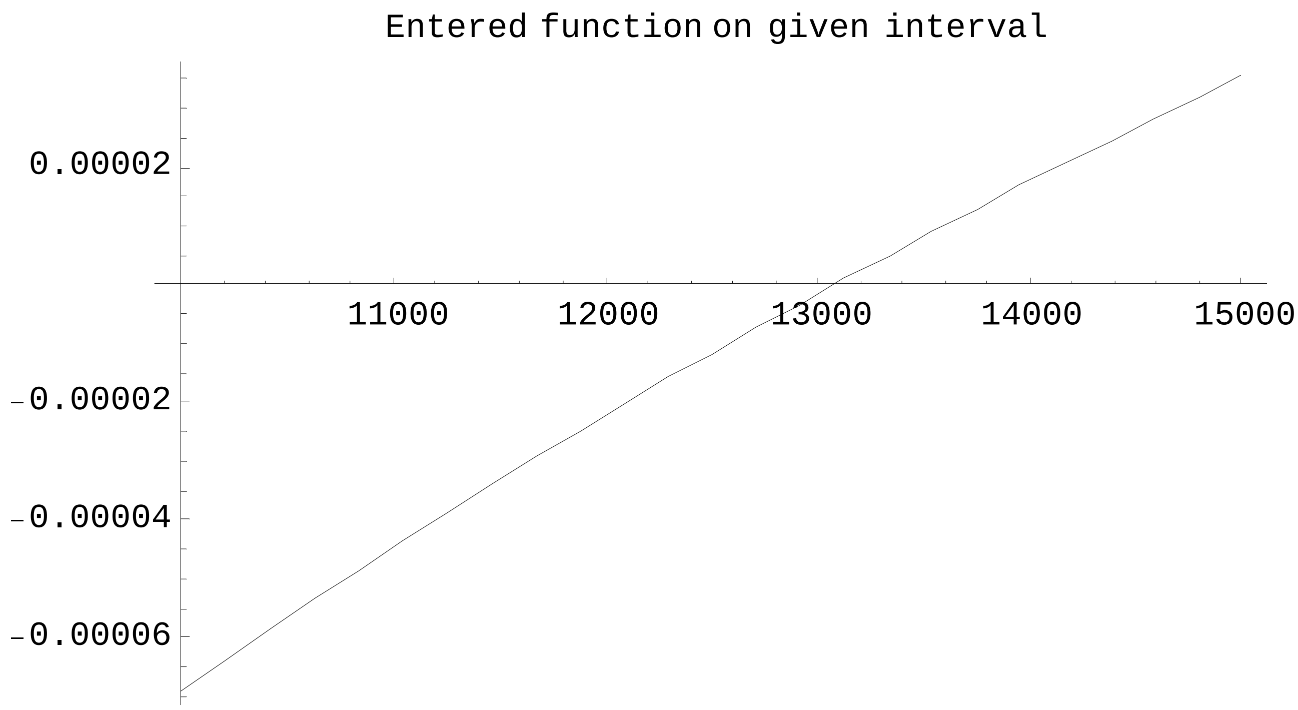
and

$$\frac{1}{18.99+273.15} = 1.129241x10^{-3} + 2.341077x10^{-4}\ln(R) + 8.775468x10^{-8}\{\ln(R)\}^{3}$$

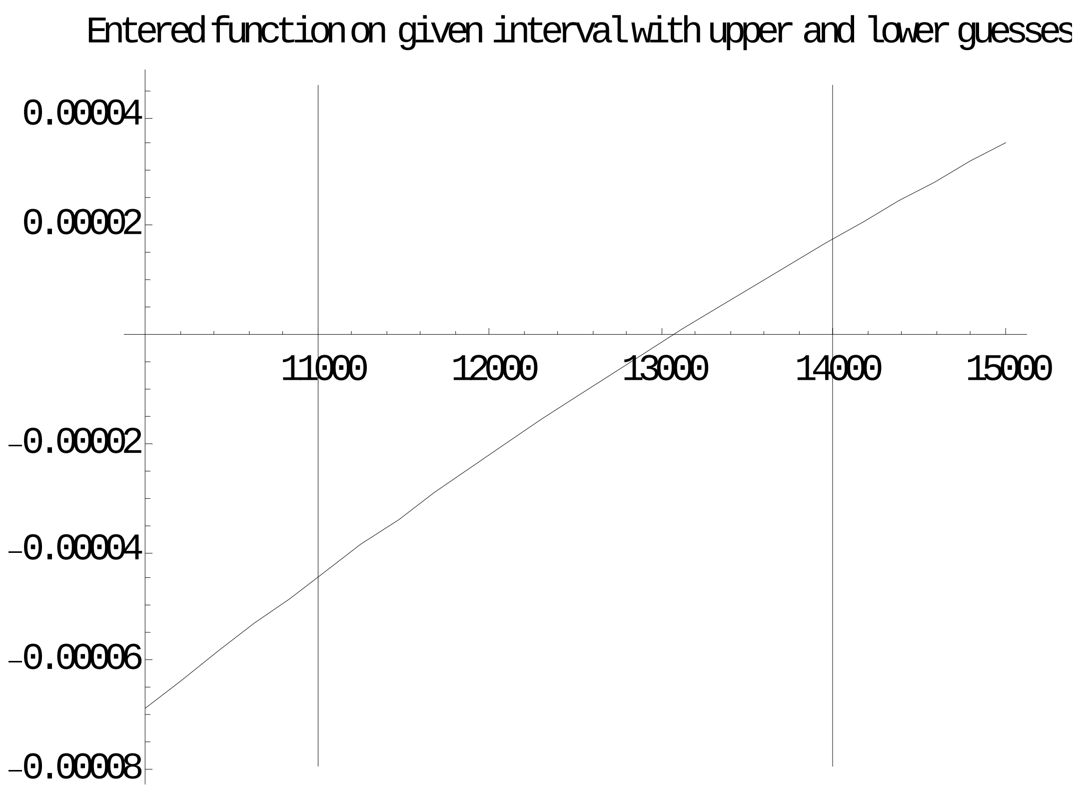
Use the bisection method of finding roots of equations to find the resistance R at 18.99^{o}C. Conduct three iterations to estimate the root of the above equation.

Graph of function *f(x)*

$$f(R) = 2.341077x10^{-4}\ln(R) + 8.775468x10^{-8}\{\ln(R)\}^3 - 2.293775x10^{-3}$$



Checking if the bracket is valid



Choose the bracket

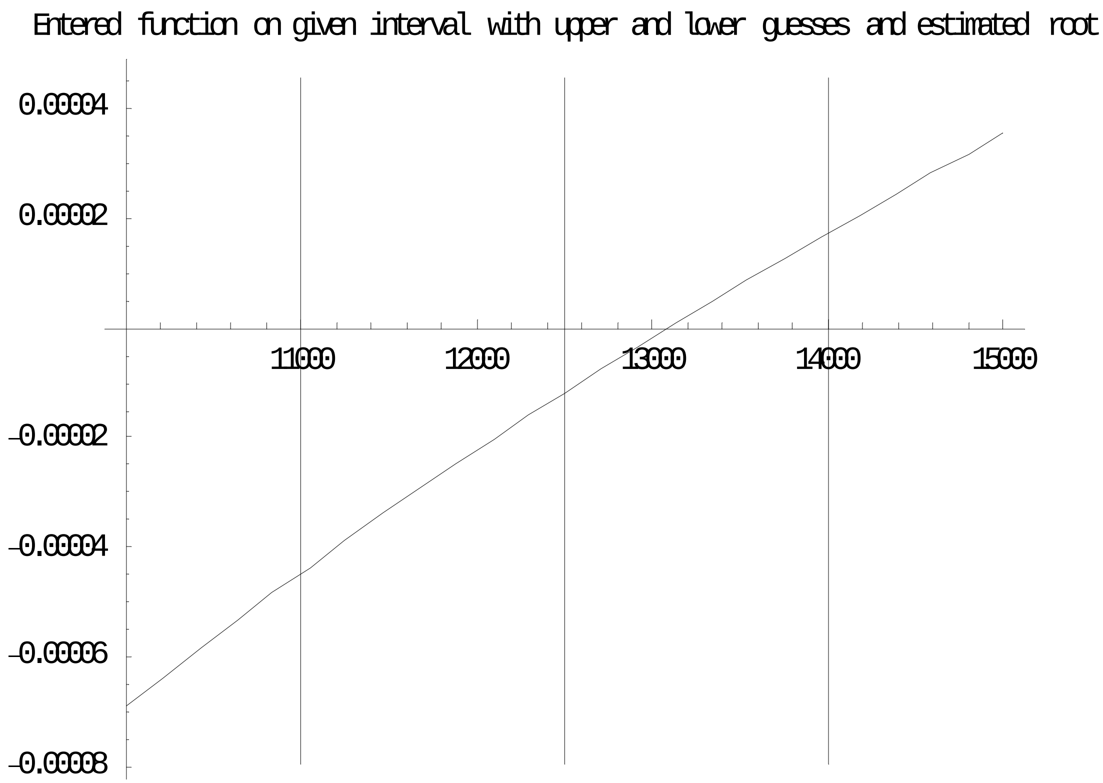
$$R_\ell = 11000$$

$$R_u = 14000$$

$$f(11000) = -4.4536 \times 10^{-5}$$

$$f(14000) = 1.7564 \times 10^{-5}$$

Iteration #1



$$R_\ell = 11000, R_u = 14000$$

$$R_m = \frac{11000 + 14000}{2} = 12500$$

$$f(11000) = -4.4536 \times 10^{-5}$$

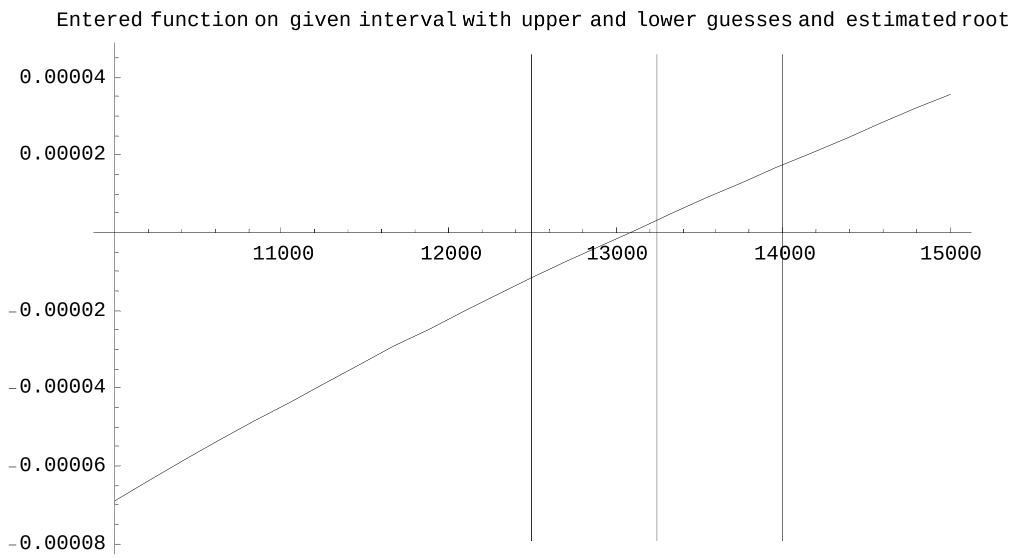
$$f(14000) = 1.7564 \times 10^{-5}$$

$$f(12500) = 1.1654 \times 10^{-5}$$

$$R_\ell = 12500$$

$$R_u = 14000$$

Iteration #2



$$R_\ell = 12500, R_u = 14000$$

$$x_m = \frac{12500 + 14000}{2} = 13250$$

$$\left|\in_a\right| = 5.6604\%$$

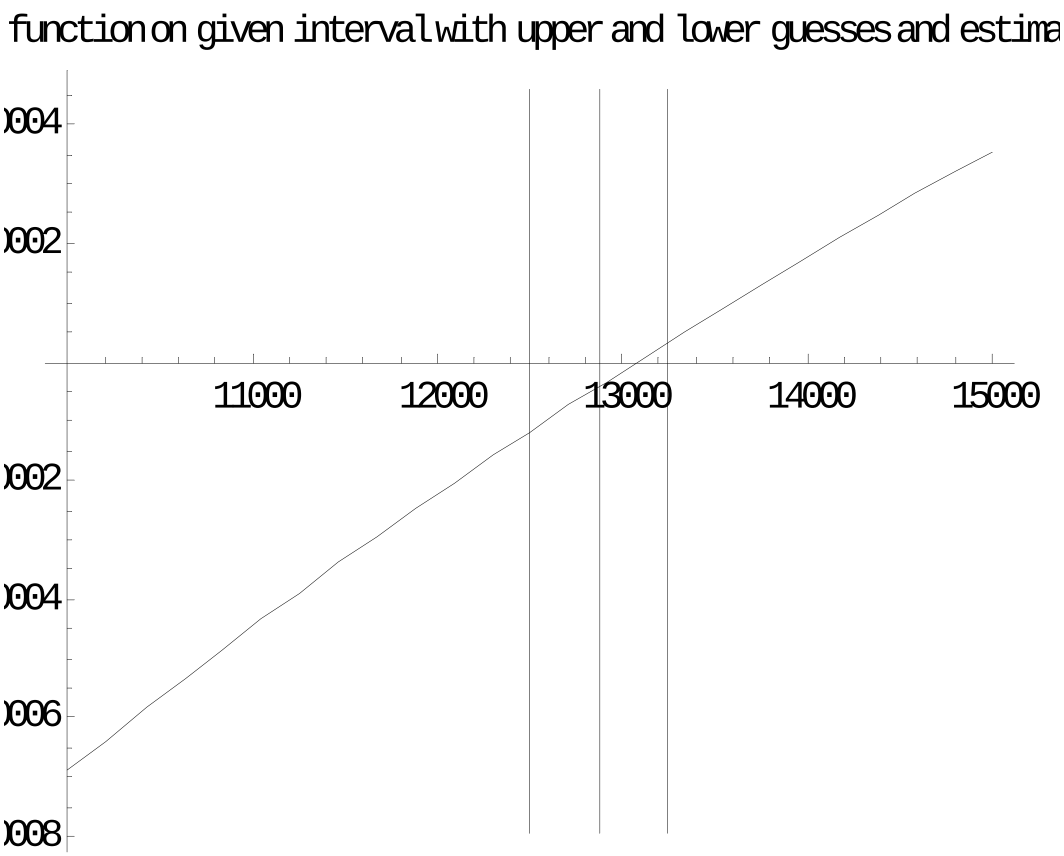
$$f(12500) = 1.1654 \times 10^{-5}$$

$$f(14000) = 1.7564 \times 10^{-5}$$

$$f(13250) = 3.3603 \times 10^{-6}$$

$$R_\ell = 12500, R_u = 13250$$

Iteration #3



$$R_\ell = 12500, R_u = 13250$$

$$R_m = \frac{12500 + 13250}{2} = 12875$$

$$|\in_a| = 2.9126\%$$

$$f(12500) = 1.1654 \times 10^{-5}$$

$$f(13250) = 3.3603 \times 10^{-6}$$

$$f(12875) = -4.0398 \times 10^{-6}$$

Convergence

Table 1: Root of $f(R)$ =0 as function of number of iterations for bisection method.

Iteration	R_l	R_u	R_m	$\lvert\epsilon_a\rvert$ %	$f(R_m)$
1	11000	14000	12500	----------	1.1654×10^{-5}
2	12500	14000	13250	5.6604	3.3603×10^{-6}
3	12500	13250	12875	2.9126	-4.0398×10^{-6}
4	12875	13250	13063	1.4354	-3.1372×10^{-7}
5	13063	13250	13156	.71259	1.5297×10^{-6}
6	13063	13156	13109	.35757	6.0961×10^{-7}
7	13063	13109	13086	.17910	1.4835×10^{-7}
8	13063	13085	13074	.089632	-8.2583×10^{-8}
9	13063	13074	13080	.044796	3.2908×10^{-8}
10	13074	13080	13077	.022403	-2.4831×10^{-8}

Advantages

- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

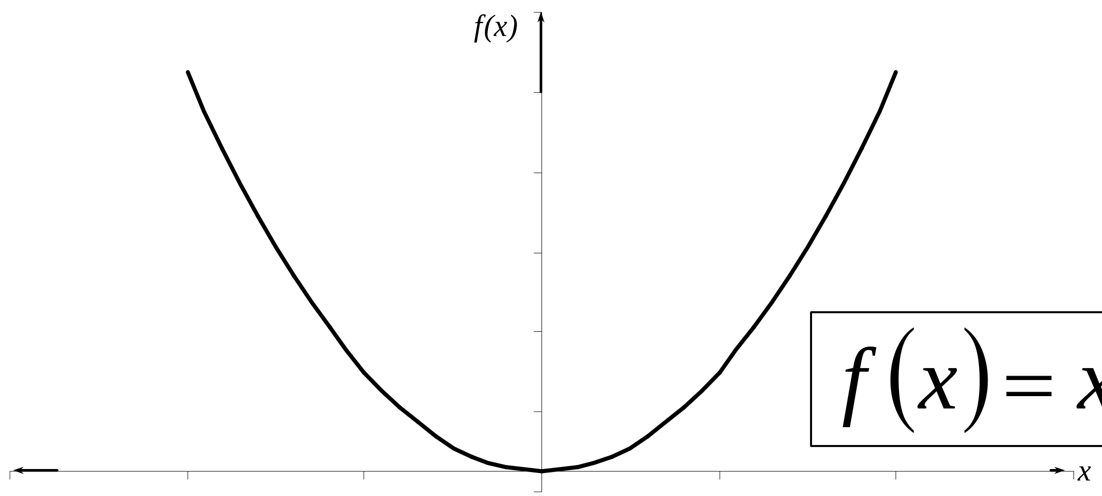
- Slow convergence

Drawbacks (continued)

- If one of the initial guesses is close to the root, the convergence is slower

Drawbacks (continued)

- If a function $f(x)$ is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



$$f(x) = x^2$$

Drawbacks (continued)

- Function changes sign but root does not exist

$$f(x) = \frac{1}{x}$$