Simultaneous Linear Equations

Holistic Numerical Methods Institute

Topic: LU Decomposition

Major: Electrical Engineering

LU Decomposition

LU Decomposition is another method to solve a set of simultaneous linear equations

Which is better, Gauss Elimination or LU Decomposition?

To answer this, a closer look at LU decomposition is needed.

LU Decomposition

Method

For most non-singular matrix $[A]$ that one could conduct Naïve Gauss Elimination forward elimination steps, one can always write it as

$$[A]=[L][U]$$

Where

$[L]$ = lower triangular martix

$[U]$ = upper triangular martix

LU Decomposition

Proof

If solving a set of linear equations $[A][X]=[C]$

If $[A]=[L][U]$ Then $[L][U][X]=[C]$

Multiply by $[L]^{-1}$

Which gives $[L]^{-1}[L][U][X]=[L]^{-1}[C]$

Remember $[L]^{-1}[L]=[I]$ which leads to $[I][U][X]=[L]^{-1}[C]$

Now, if $[I][U]=[U]$ then $[U][X]=[L]^{-1}[C]$

Now, let $[L]^{-1}[C]=[Z]$

Which ends with $[L][\mathrm{Z}]=[C]$ (1)

and $[U][\mathrm{X}]=[Z]$ (2)

LU Decomposition

How can this be used?

Given $[A][X]=[C]$

Decompose $[\mathrm{A}]$ into $[L]$ and $[U]$

Then solve $[L][Z]=[\mathrm{C}]$ for $[Z]$

And then solve $[U][\mathrm{X}]=[Z]$ for $[X]$

LU Decomposition

How is this better or faster than Gauss Elimination?

Let's look at computational time.

n = number of equations

To decompose [A], time is proportional to $\frac{n^3}{3}$

To solve $[U][X]=[C]$ and $[L][Z]=[C]$

time proportional to $\frac{n^2}{2}$

LU Decomposition

Therefore, total computational time for LU Decomposition is proportional to

$$\frac{n^3}{3} + 2\left(\frac{n^2}{2}\right) \quad \text{or} \quad \frac{n^3}{3} + n^2$$

Gauss Elimination computation time is proportional to

$$\frac{n^3}{3} + \frac{n^2}{2}$$

How is this better?

LU Decomposition

What about a situation where the [C] vector changes?

In LU Decomposition, LU decomposition of [A] is independent of the [C] vector, therefore it only needs to be done once.

Let m = the number of times the [C] vector changes

The computational times are proportional to

LU decomposition = $m(\frac{n^3}{3}+\frac{n^2}{2})$ Gauss Elimination= $\frac{n^3}{3}+m(n^2)$

Consider a 100 equation set with 50 right hand side vectors

LU Decomposition = 8.33×10^5 Gauss Elimination = 1.69×10^7

LU Decomposition

Another Advantage

Finding the Inverse of a Matrix

LU Decomposition

$$\frac{n^3}{3} + n(n^2) = \frac{4n^3}{3}$$

Gauss Elimination

$$n\left(\frac{n^3}{3} + \frac{n^2}{2}\right) = \frac{n^4}{3} + \frac{n^3}{2}$$

For large values of n

$$\frac{n^4}{3} + \frac{n^3}{2} \rangle\rangle \frac{4n^3}{3}$$

LU Decomposition

Method: [A] Decompose to [L] and [U]

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ \ell_{21} & 1 & 0\\ \ell_{31} & \ell_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

[U] is the same as the coefficient matrix at the end of the forward elimination step.

[L] is obtained using the *multipliers* that were used in the forward elimination process

Example: Unbalanced three phase load

Three-phase loads are common in AC systems. When the system is balanced the analysis can be simplified to a single equivalent circuit model. However, when it is unbalanced the only practical solution involves the solution of simultaneous linear equations. In a model the following equations need to be solved.

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix} \begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 120 \\ 0.000 \\ -60.00 \\ -103.9 \\ -60.00 \\ 103.9 \end{bmatrix}$$

Find the values of I_{ar}, I_{ai}, I_{br}, I_{bi}, I_{cr}, and I_{ci} using LU Decomposition.

Example: Unbalanced three phase load

Finding the [*U*] matrix

Using the Forward Elimination Procedure of Gauss Elimination to obtain the [*U*] matrix from the initial coefficient matrix.

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix}$$

Example: Unbalanced three phase load

Forward Elimination: Step 1

For the new row 2: $\text{Row}2-\left[\dfrac{\text{Row}1}{0.7460}\right]\times(0.4516)=$

$$\begin{bmatrix}0 & 1.019381 & 0.001946 & 0.014843 & 0.001946 & 0.014843\end{bmatrix}$$

For the new row 3: $\text{Row}3-\left[\dfrac{\text{Row}1}{0.7460}\right]\times(0.0100)=$

$$\begin{bmatrix}0 & -0.001946 & 0.778566 & -0.520393 & 0.009866 & -0.007893\end{bmatrix}$$

Example: Unbalanced three phase load

Forward Elimination: Step 1

For the new row 4:

$$\mathrm{Row4} - \left[\frac{\mathrm{Row1}}{0.7460}\right] \times (0.0080) =$$

$$\begin{bmatrix} 0 & 0.014843 & 0.520393 & 0.778786 & 0.007893 & 0.010086 \end{bmatrix}$$

For the new row 5:

$$\mathrm{Row5} - \left[\frac{\mathrm{Row1}}{0.7460}\right] \times (0.0100) =$$

$$\begin{bmatrix} 0 & -0.001946 & 0.009866 & -0.007893 & 0.807866 & -0.603893 \end{bmatrix}$$

Example: Unbalanced three phase load

Forward Elimination: Step 1

For the new row 6:

$$\text{Row}6 - \left[\frac{\text{Row}1}{0.7460}\right] \times (0.0080) =$$

$$\begin{bmatrix} 0 & 0.014843 & 0.007893 & 0.010086 & 0.603893 & 0.808086 \end{bmatrix}$$

The system of equations after the completion of the first step of forward elimination is:

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.019381 & 0.001946 & 0.014843 & 0.001946 & 0.014843 \\ 0 & -0.001946 & 0.778566 & -0.520393 & 0.009866 & -0.007893 \\ 0 & 0.014843 & 0.520393 & 0.778786 & 0.007893 & 0.010086 \\ 0 & -0.001946 & 0.009866 & -0.007893 & 0.807866 & -0.603893 \\ 0 & 0.014843 & 0.007893 & 0.010086 & 0.603893 & 0.808086 \end{bmatrix}$$

Example: Unbalanced three phase load

Forward Elimination: Step 2

For the new row 3:

$$\text{Row3} - \left[\frac{\text{Row2}}{1.019381}\right] \times (-0.001946) =$$

$$\begin{bmatrix} 0 & 0 & 0.77857 & -0.520364 & 0.00987 & -0.007864 \end{bmatrix}$$

For the new row 4:

$$\text{Row4} - \left[\frac{\text{Row2}}{1.019381}\right] \times (0.014843) =$$

$$\begin{bmatrix} 0 & 0 & 0.520364 & 0.77857 & 0.007864 & 0.00987 \end{bmatrix}$$

Example: Unbalanced three phase load

Forward Elimination: Step 2

<u>For the new row 5:</u>

$$\text{Row}5 - \left[\frac{\text{Row}2}{1.019381}\right] \times (-0.001946) =$$

$$\begin{bmatrix}0 & 0 & 0.00987 & -0.007864 & 0.80787 & -0.603864\end{bmatrix}$$

<u>For the new row 6:</u>

$$\text{Row}6 - \left[\frac{\text{Row}2}{1.019381}\right] \times (0.014843) =$$

$$\begin{bmatrix}0 & 0 & 0.007864 & 0.00987 & 0.603864 & 0.80787\end{bmatrix}$$

Example: Unbalanced three phase load

The system of equations after the completion of the second step of forward elimination is:

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.019381 & 0.001946 & 0.014843 & 0.001946 & 0.014843 \\ 0 & 0 & 0.77857 & -0.520364 & 0.00987 & -0.007864 \\ 0 & 0 & 0.520364 & 0.77857 & 0.007864 & 0.00987 \\ 0 & 0 & 0.00987 & -0.007864 & 0.80787 & -0.603864 \\ 0 & 0 & 0.007864 & 0.00987 & 0.603864 & 0.80787 \end{bmatrix}$$

Example: Unbalanced three phase load

Forward Elimination: Step 3

<u>For the new row 4:</u>

$$\text{Row}4 - \left[\frac{\text{Row}3}{0.77857}\right] \times (0.520364) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 1.12636 & 0.001268 & 0.015126 \end{bmatrix}$$

<u>For the new row 5:</u>

$$\text{Row}5 - \left[\frac{\text{Row}3}{0.77857}\right] \times (0.00987) =$$

$$\begin{bmatrix} 0 & 0 & 0 & -0.001268 & 0.807745 & -0.603765 \end{bmatrix}$$

Example: Unbalanced three phase load

Forward Elimination: Step 3

For the new row 6: $\text{Row6} - \left[\dfrac{\text{Row3}}{0.77857}\right] \times (0.007864) =$

$$\begin{bmatrix} 0 & 0 & 0 & 0.015126 & 0.603765 & 0.807949 \end{bmatrix}$$

The system of equations after the completion of the third step of forward elimination is:

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.019381 & 0.001946 & 0.014843 & 0.001946 & 0.014843 \\ 0 & 0 & 0.77857 & -0.520364 & 0.00987 & -0.007864 \\ 0 & 0 & 0 & 1.12636 & 0.001268 & 0.015126 \\ 0 & 0 & 0 & -0.001268 & 0.807745 & -0.603765 \\ 0 & 0 & 0 & 0.015126 & 0.603765 & 0.807949 \end{bmatrix}$$

Example: Unbalanced three phase load

Forward Elimination: Step 4

For the new row 5:

$$\text{Row}5-\left[\frac{\text{Row}4}{1.12636}\right]\times(-0.001268)=$$

$$\begin{bmatrix}0 & 0 & 0 & 0 & 0.807746 & -0.603748\end{bmatrix}$$

For the new row 6:

$$\text{Row}6-\left[\frac{\text{Row}4}{1.12636}\right]\times(0.015126)=$$

$$\begin{bmatrix}0 & 0 & 0 & 0 & 0.603748 & 0.807746\end{bmatrix}$$

Example: Unbalanced three phase load

The system of equations after the completion of the fourth step of forward elimination is:

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.019381 & 0.001946 & 0.014843 & 0.001946 & 0.014843 \\ 0 & 0 & 0.77857 & -0.520364 & 0.00987 & -0.007864 \\ 0 & 0 & 0 & 1.12636 & 0.001268 & 0.015126 \\ 0 & 0 & 0 & 0 & 0.807746 & -0.603748 \\ 0 & 0 & 0 & 0 & 0.603748 & 0.807746 \end{bmatrix}$$

Example: Unbalanced three phase load

Forward Elimination: Step 5

For the new row 6:

$$\text{Row6} - \left[\frac{\text{Row5}}{0.807746}\right] \times (0.603748) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1.259016 \end{bmatrix}$$

The coefficient matrix at the end of the forward elimination process is the [*U*] matrix

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.019381 & 0.001946 & 0.014843 & 0.001946 & 0.014843 \\ 0 & 0 & 0.77857 & -0.520364 & 0.00987 & -0.007864 \\ 0 & 0 & 0 & 1.12636 & 0.001268 & 0.015126 \\ 0 & 0 & 0 & 0 & 0.807746 & -0.603748 \\ 0 & 0 & 0 & 0 & 0 & 1.259016 \end{bmatrix}$$

Example: Unbalanced three phase load

Finding the [*L*] matrix

For a system of six equations, the [*L*] matrix is in the form

$$[L]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 & 0 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 & 0 & 0 \\ \ell_{51} & \ell_{52} & \ell_{53} & \ell_{54} & 1 & 0 \\ \ell_{61} & \ell_{62} & \ell_{63} & \ell_{64} & \ell_{65} & 1 \end{bmatrix}$$

Values of the [*L*] matrix are the multipliers used during the Forward Elimination Procedure

Example: Unbalanced three phase load

Finding the [*L*] matrix

From the first step of forward elimination

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix}$$

$$\ell_{21} = \frac{0.4516}{0.7460} = 0.605362$$

$$\ell_{31} = \frac{0.01}{0.7460} = 0.013405$$

$$\ell_{41} = \frac{0.008}{0.7460} = 0.010724$$

$$\ell_{51} = \frac{0.01}{0.7460} = 0.013405$$

$$\ell_{61} = \frac{0.008}{0.7460} = 0.010724$$

Example: Unbalanced three phase load

Finding the [*L*] matrix

From the second step of forward elimination

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.019381 & 0.001946 & 0.014843 & 0.001946 & 0.014843 \\ 0 & -0.001946 & 0.778566 & -0.520393 & 0.009866 & -0.007893 \\ 0 & 0.014843 & 0.520393 & 0.778786 & 0.007893 & 0.010086 \\ 0 & -0.001946 & 0.009866 & -0.007893 & 0.807866 & -0.603893 \\ 0 & 0.014843 & 0.007893 & 0.010086 & 0.603893 & 0.808086 \end{bmatrix}$$

$$\ell_{32} = \frac{-0.001946}{1.019381} = -0.001909$$

$$\ell_{42} = \frac{0.014843}{1.019381} = 0.014561$$

$$\ell_{52} = \frac{-0.001946}{1.019381} = -0.001909$$

$$\ell_{62} = \frac{0.014843}{1.019381} = 0.014561$$

Example: Unbalanced three phase load

Finding the [*L*] matrix

From the third step of forward elimination

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.019381 & 0.001946 & 0.014843 & 0.001946 & 0.014843 \\ 0 & 0 & 0.77857 & -0.520364 & 0.00987 & -0.007864 \\ 0 & 0 & 0.520364 & 0.77857 & 0.007864 & 0.00987 \\ 0 & 0 & 0.00987 & -0.007864 & 0.80787 & -0.603864 \\ 0 & 0 & 0.007864 & 0.00987 & 0.603864 & 0.80787 \end{bmatrix}$$

$$\ell_{43} = \frac{0.520364}{0.77857} = 0.668359$$

$$\ell_{53} = \frac{0.00987}{0.77857} = 0.012677$$

$$\ell_{63} = \frac{0.007864}{0.77857} = 0.010101$$

Example: Unbalanced three phase load

Finding the [*L*] matrix

From the fourth step of forward elimination

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.019381 & 0.001946 & 0.014843 & 0.001946 & 0.014843 \\ 0 & 0 & 0.77857 & -0.520364 & 0.00987 & -0.007864 \\ 0 & 0 & 0 & 1.12636 & 0.001268 & 0.015126 \\ 0 & 0 & 0 & -0.001268 & 0.807745 & -0.603765 \\ 0 & 0 & 0 & 0.015126 & 0.603765 & 0.807949 \end{bmatrix}$$

$$\ell_{54} = \frac{-0.001268}{1.12636} = -0.001126$$

$$\ell_{64} = \frac{0.015126}{1.12636} = 0.013429$$

Example: Unbalanced three phase load

Finding the [*L*] matrix

From the fifth step of forward elimination

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.019381 & 0.001946 & 0.014843 & 0.001946 & 0.014843 \\ 0 & 0 & 0.77857 & -0.520364 & 0.00987 & -0.007864 \\ 0 & 0 & 0 & 1.12636 & 0.001268 & 0.015126 \\ 0 & 0 & 0 & 0 & 0.807746 & -0.603748 \\ 0 & 0 & 0 & 0 & 0.603748 & 0.807746 \end{bmatrix}$$

$$\ell_{65} = \frac{0.603748}{0.807746} = 0.747447$$

Example: Unbalanced three phase load

The [*L*] matrix is

$$[L]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.605362 & 1 & 0 & 0 & 0 & 0 \\ 0.013405 & -0.001909 & 1 & 0 & 0 & 0 \\ 0.010724 & 0.014561 & 0.668359 & 1 & 0 & 0 \\ 0.013405 & -0.001909 & 0.012677 & -0.001126 & 1 & 0 \\ 0.010724 & 0.014561 & 0.010101 & 0.013429 & 0.747447 & 1 \end{bmatrix}$$

Now find the [*Z*] matrix using [*L*] [*Z*] = [*C*]

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.605362 & 1 & 0 & 0 & 0 & 0 \\ 0.013405 & -0.001909 & 1 & 0 & 0 & 0 \\ 0.010724 & 0.014561 & 0.668359 & 1 & 0 & 0 \\ 0.013405 & -0.001909 & 0.012677 & -0.001126 & 1 & 0 \\ 0.010724 & 0.014561 & 0.010101 & 0.013429 & 0.747447 & 1 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \end{bmatrix}=\begin{bmatrix} 120 \\ 0.000 \\ -60.00 \\ -103.9 \\ -60.00 \\ 103.9 \end{bmatrix}$$

Example: Unbalanced three phase load

From [*L*] [*Z*] = [*C*], these six equations are obtained

$$z_1 = 120$$

$$0.605362 z_1 + z_2 = 0.00$$

$$0.013405 z_1 + (-0.001909) z_2 + z_3 = -60.00$$

$$0.010724 z_1 + 0.014561 z_2 + 0.668359 z_3 + z_4 = -103.9$$

$$0.013405 z_1 + (-0.001909) z_2 + 0.012677 z_3 + (-0.001126) z_4 + z_5 = -60.00$$

$$0.010724 z_1 + 0.014561 z_2 + 0.010101 z_3 + 0.013429 z_4 + 0747447 z_5 + z_6 = 103.9$$

Example: Unbalanced three phase load

Complete the forward substitution to solve for [*Z*]

Forward substitution starting from the first equation gives

$$z_1 = 120$$

Substituting the value of z_1 into the second equation

$$z_2 = 0.00 - 0.605362 z_1$$

$$= -72.64343$$

Substituting the value of z_1 and z_2 into the third equation

$$z_3 = -60.00 - 0.013405 z_1 - (-0.001909) z_2$$

$$= -61.74728$$

Example: Unbalanced three phase load

Complete the forward substitution to solve for [Z]

Substituting the value of z_1, z_2, and z_3 into the fourth equation

$$z_4 = -103.9 - 0.010724z_1 - 0.014561z_2 - 0.668359z_3$$
$$= -62.85974$$

Substituting the value of z_1, z_2, z_3, and z_4 into the fifth equation

$$z_5 = -60.00 - 0.013405z_1 - (-0.001909)z_2 - 0.012677z_3 - (-0.001126)z_4$$
$$= -61.03529$$

Substituting the value of z_1, z_2, z_3, z_4, and z_5 into the sixth equation

$$z_6 = 103.9 - 0.010724z_1 - 0.014561z_2 - 0.010101z_3 - 0.013429z_4 - 0747447z_5$$
$$= 150.7594$$

Example: Unbalanced three phase load

The [*Z*] matrix is $[Z]=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix}=\begin{bmatrix} 120 \\ -72.64343 \\ -61.74728 \\ -62.85974 \\ -61.03529 \\ 150.7594 \end{bmatrix}$

Now set [*U*] [*X*] = [*Z*] and solve for [*X*]

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.019381 & 0.001946 & 0.014843 & 0.001946 & 0.014843 \\ 0 & 0 & 0.77857 & -0.520364 & 0.00987 & -0.007864 \\ 0 & 0 & 0 & 1.12636 & 0.001268 & 0.015126 \\ 0 & 0 & 0 & 0 & 0.807746 & -0.603748 \\ 0 & 0 & 0 & 0 & 0 & 1.259016 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}=\begin{bmatrix} 120 \\ -72.64343 \\ -61.74728 \\ -62.85974 \\ -61.03529 \\ 150.7594 \end{bmatrix}$$

Example: Unbalanced three phase load

From [*U*] [*X*] = [*Z*] , we obtain the following six equations

$$0.7460x_1 + (-0.4516)x_2 + 0.0100x_3 + (-0.0080)x_4 + 0.0100x_5 + \left(-0.0080\right)x_6 = 120$$

$$1.019381x_2 + 0.001946x_3 + 0.014843x_4 + 0.001946x_5 + 0.014843x_6 = -72.64343$$

$$0.77857x_3 + (-0.520364)x_4 + 0.00987x_5 + (-0.007864)x_6 = -61.74728$$

$$1.12636x_4 + 0.001268x_5 + 0.015126x_6 = -62.85974$$

$$0.807746x_5 + (-0.603748)x_6 = -61.03529$$

$$1.259016x_6 = 150.7594$$

Example: Unbalanced three phase load

Solve the six equations, starting with the sixth equation and back substitute each new value each equation.

From the sixth equation

$$1.259016 x_6 = 150.7594$$

$$x_6 = \frac{150.7594}{1.259016} = 119.7439$$

Substituting the value of x_6 into the fifth equation

$$0.807746 x_5 + (-0.603748) x_6 = -61.03529$$

$$x_5 = \frac{-61.03529 - (-0.603748) x_6}{0.807746} = 13.93977$$

Example: Unbalanced three phase load

Substituting the values of x_5 and x_6 into the fourth equation

$$1.12636x_4 + 0.001268x_5 + 0.015126x_6 = -62.85974$$

$$x_4 = \frac{-62.85974 - 0.001268x_5 - 0.015126x_6}{1.12636} = -57.43159$$

Substituting the values of x_4, x_5, and x_6 into the third equation

$$0.77857x_3 + (-0.520364)x_4 + 0.00987x_5 + (-0.007864)x_6 = -61.74728$$

$$x_3 = \frac{-61.74728 - (-0.520364)x_4 - 0.00987x_5 - (-0.007864)x_6}{0.77857} = -116.6607$$

Example: Unbalanced three phase load

Substituting the values of x_3, x_4, x_5, and x_6 into the second equation

$$1.019381x_2 + 0.001946x_3 + 0.014843x_4 + 0.001946x_5 + 0.014843x_6 = -72.64343$$

$$x_2 = \frac{-72.64343 - 0.001946x_3 - 0.014843x_4 - 0.001946x_5 - 0.014843x_6}{1.019381} = -71.97344$$

Substituting the values of x_2, x_3, x_4, x_5, and x_6 into the first equation

$$0.7460x_1 + (-0.4516)x_2 + 0.0100x_3 + (-0.0080)x_4 + 0.0100x_5 + \left(-0.0080\right)x_6 = 120$$

$$x_1 = \frac{120 - (-0.4516)x_2 - 0.0100x_3 - (-0.0080)x_4 - 0.0100x_5 - \left(-0.0080\right)x_6}{0.7460} = 119.3331$$

Example: Unbalanced three phase load

Solution:

The solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 119.3331 \\ -71.97344 \\ -116.6607 \\ -57.43159 \\ 13.93977 \\ 119.7439 \end{bmatrix}$$

LU Decomposition

Finding the inverse of a square matrix

Remember, the relative computational time comparison of LU decomposition and Gauss elimination is:

$$\frac{n^4}{3} + \frac{n^3}{2} \rangle\rangle \frac{4n^3}{3}$$

Review: The inverse [B] of a square matrix [*A*] is defined as

$$[A][B] = [I] = [B][A]$$

LU Decomposition

Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse?

Assume the first column of [*B*] to be $\begin{bmatrix} b_{11} & b_{12} & \ldots & b_{n1} \end{bmatrix}^T$

Using this and the definition of matrix multiplication

First column of [*B*]

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second column of [*B*]

$$[A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in [*B*] can be found in the same manner

LU Decomposition

Example: Finding the inverse of a square matrix

Find the inverse of [*A*]

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$$

Using the Decomposition procedure, the [*L*] and [*U*] matrices are found to be

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}$$

LU Decomposition

Example: Finding the inverse of a square matrix

Solving for the each column of [*B*] requires to steps

1) Solve [*L*] [*Z*] = [*C*] for [*Z*] and 2) Solve [*U*] [*X*] = [*Z*] for [*X*]

Step 1: $[\mathrm{L}][\mathrm{Z}]=[\mathrm{C}]\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

This generates the equations:

$$z_1 = 1$$

$$2.56z_1 + z_2 = 0$$

$$5.76z_1 + 3.5z_2 + z_3 = 0$$

LU Decomposition

Example: Finding the inverse of a square matrix

Solving for [*Z*]

$$z_1 = 1$$

$$\begin{aligned} z_2 &= 0 - 2.56 z_1 \\ &= 0 - 2.56(1) \\ &= -2.56 \end{aligned}$$

$$\begin{aligned} z_3 &= 0 - 5.76 z_1 - 3.5 z_2 \\ &= 0 - 5.76(1) - 3.5(-2.56) \\ &= 3.2 \end{aligned}$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

LU Decomposition

Example: Finding the inverse of a square matrix

Solving for [U] [X] = [Z] for [X]

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$

$$-4.8b_{21} - 1.56b_{31} = -2.56$$

$$0.7b_{31} = 3.2$$

LU Decomposition

Example: Finding the inverse of a square matrix

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560 b_{31}}{-4.8}$$

$$= \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25}$$

$$= \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762$$

So the first column of the inverse of [*A*] is:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

LU Decomposition

Example: Finding the inverse of a square matrix

Repeating for the second and third columns of the inverse

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} \mathrm{b}_{13} \\ \mathrm{b}_{23} \\ \mathrm{b}_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

LU Decomposition

Example: Finding the inverse of a square matrix

The inverse of [*A*] is

$$[A]^{-1} = \begin{bmatrix} 0.4762 & 0.08333 & 0.0357 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.050 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$[\mathrm{A}][\mathrm{A}]^{-1} = [\mathrm{I}] = [\mathrm{A}]^{-1}[\mathrm{A}]$$