Simultaneous Linear Equations

Holistic
Numerical
Methods
Institute

Topic: Gauss-Seidel Method

Major: Electrical Engineering

Gauss-Seidel Method

An *iterative* method.

Basic Procedure:

-Algebraically solve each linear equation for x_i

-Assume an initial guess solution array

-Solve for each x_i and repeat

-Use absolute relative approximate error after each iteration to check if error is within a pre-specified tolerance.

Gauss-Seidel Method

Why?

The Gauss-Seidel Method allows the user to control round-off error.

Elimination methods such as Gaussian Elimination and LU Decomposition are prone to prone to round-off error.

Also: If the physics of the problem are understood, a close initial guess can be made, decreasing the number of iterations needed.

Gauss-Seidel Method

Algorithm

A set of *n* equations and *n* unknowns:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\begin{matrix} \vdots & & \vdots \end{matrix}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

If: the diagonal elements are non-zero

Rewrite each equation solving for the corresponding unknown

ex:

First equation, solve for x_1

Second equation, solve for x_2

Gauss-Seidel Method

Algorithm

Rewriting each equation

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$

From Equation 1

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

From equation 2

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

From equation n-1

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

From equation n

Gauss-Seidel Method

Algorithm

General Form of each equation

$$x_1 = \frac{c_1 - \sum\limits_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j} x_j}{a_{11}}$$

$$x_{n-1} = \frac{c_{n-1} - \sum\limits_{\substack{j=1 \\ j\neq n-1}}^{n} a_{n-1,j} x_j}{a_{n-1,n-1}}$$

$$x_2 = \frac{c_2 - \sum\limits_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j} x_j}{a_{22}}$$

$$x_n = \frac{c_n - \sum\limits_{\substack{j=1 \\ j\neq n}}^{n} a_{nj} x_j}{a_{nn}}$$

Gauss-Seidel Method

Algorithm

General Form for any row ‘i’

$$x_i = \frac{c_i - \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, i = 1, 2, \ldots, n.$$

How or where can this equation be used?

Gauss-Seidel Method

Solve for the unknowns

Assume an initial guess for [X]

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Use rewritten equations to solve for each value of xi.

Important: Remember to use the most recent value of x_i. Which means to apply values calculated to the calculations remaining in the **current** iteration.

Gauss-Seidel Method

Calculate the Absolute Relative Approximate Error

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

So when has the answer been found?

The iterations are stopped when the absolute relative approximate error is less than a pre-specified tolerance for all unknowns.

Example: Unbalanced three phase load

Three-phase loads are common in AC systems. When the system is balanced the analysis can be simplified to a single equivalent circuit model. However, when it is unbalanced the only practical solution involves the solution of simultaneous linear equations. In a model the following equations need to be solved.

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix} \begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 120 \\ 0.000 \\ -60.00 \\ -103.9 \\ -60.00 \\ 103.9 \end{bmatrix}$$

Find the values of I_{ar}, I_{ai}, I_{br}, I_{bi}, I_{cr}, and I_{ci} using the Gauss-Seidel method.

Example: Unbalanced three phase load

Rewrite each equation to solve for each of the unknowns

$$I_{ar} = \frac{120.00 - (-0.4516)I_{ai} - 0.0100 I_{br} - (-0.0080)I_{bi} - 0.0100 I_{cr} - (-0.0080)I_{ci}}{0.7460}$$

$$I_{ai} = \frac{0.00 - 0.4516 I_{ar} - 0.0080 I_{br} - 0.0100 I_{bi} - 0.0080 I_{cr} - 0.0100 I_{ci}}{0.7460}$$

$$I_{br} = \frac{-60.00 - 0.0100 I_{ai} - (-0.0080)I_{ar} - (-0.5205)I_{bi} - 0.0100 I_{cr} - (-0.0080)I_{ci}}{0.7787}$$

$$I_{bi} = \frac{-103.9 - 0.0080 I_{ar} - 0.0100 I_{ai} - 0.5205 I_{br} - 0.0080 I_{cr} - 0.0100 I_{ci}}{0.7787}$$

$$I_{cr} = \frac{-60.00 - 0.0100 I_{ar} - (-0.0080)I_{ai} - 0.0100 I_{br} - (-0.0080)I_{bi} - (-0.6040)I_{ci}}{0.8080}$$

$$I_{ci} = \frac{103.9 - 0.0080 I_{ar} - 0.0100 I_{ai} - 0.0080 I_{br} - 0.0100 I_{bi} - 0.6040 I_{cr}}{0.8080}$$

Example: Unbalanced three phase load

For iteration #1, start with an initial guess value

Initial Guess:

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{bmatrix}$$

Example: Unbalanced three phase load

Substituting the guess values into the first equation

$$I_{ar} = \frac{120.00 - (-0.4516) I_{ai} - 0.0100 I_{br} - (-0.0080) I_{bi} - 0.0100 I_{cr} - (-0.0080) I_{ci}}{0.7460}$$

$$I_{ar} = 172.8579$$

Substituting the new value of I_{ar} and the remaining guess values into the second equation

$$I_{ai} = \frac{0.00 - 0.4516 I_{ar} - 0.0080 I_{br} - 0.0100 I_{bi} - 0.0080 I_{cr} - 0.0100 I_{ci}}{0.7460}$$

$$I_{ai} = -105.6067$$

Example: Unbalanced three phase load

Substituting the new values I_{ar} , I_{ai} , and the remaining guess values into the third equation

$$I_{br} = \frac{-60.00 - 0.0100 I_{ai} - (-0.0080) I_{ar} - (-0.5205) I_{bi} - 0.0100 I_{cr} - (-0.0080) I_{ci}}{0.7787}$$

$$I_{br} = -67.0392$$

Substituting the new values I_{ar} , I_{ai} , I_{br} , and the remaining guess values into the fourth equation

$$I_{bi} = \frac{-103.9 - 0.0080 I_{ar} - 0.0100 I_{ai} - 0.5205 I_{br} - 0.0080 I_{cr} - 0.0100 I_{ci}}{0.7787}$$

$$I_{bi} = -89.4990$$

Example: Unbalanced three phase load

Substituting the new values I_{ar}, I_{ai}, I_{br}, I_{bi}, and the remaining guess values into the fifth equation

$$I_{cr} = \frac{-60.00 - 0.0100 I_{ar} - (-0.0080) I_{ai} - 0.0100 I_{br} - (-0.0080) I_{bi} - (-0.6040) I_{ci}}{0.8080}$$

$$I_{cr} = -62.5483$$

Substituting the new values I_{ar}, I_{ai}, I_{br}, I_{bi}, I_{cr}, and the remaining guess value into the sixth equation

$$I_{ci} = \frac{103.9 - 0.0080 I_{ar} - 0.0100 I_{ai} - 0.0080 I_{br} - 0.0100 I_{bi} - 0.6040 I_{cr}}{0.8080}$$

$$I_{ci} = 176.7125$$

Example: Unbalanced three phase load

At the end of the first iteration, the solution matrix is:

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 172.8579 \\ -105.6067 \\ -67.0392 \\ -89.4990 \\ -62.5483 \\ 176.7125 \end{bmatrix}$$

How accurate is the solution? Find the absolute relative approximate error using:

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}}\right| \times 100$$

Example: Unbalanced three phase load

Calculating the absolute relative approximate errors

$$\left|\varepsilon_a\right|_1 = \left|\frac{172.8579 - 20}{172.8579}\right| \times 100 = 73.5522\%$$

$$\left|\varepsilon_a\right|_2 = \left|\frac{-105.6067 - 20}{-105.6067}\right| \times 100 = 83.0126\%$$

$$\left|\varepsilon_a\right|_3 = \left|\frac{-67.0392 - 20}{-67.0392}\right| \times 100 = 51.4009\%$$

$$\left|\varepsilon_a\right|_4 = \left|\frac{-89.4990 - 20}{-89.4990}\right| \times 100 = 107.4993\%$$

$$\left|\varepsilon_a\right|_5 = \left|\frac{-62.5483 - 20}{-62.5483}\right| \times 100 = 209.1295\%$$

$$\left|\varepsilon_a\right|_6 = \left|\frac{176.7125 - 20}{176.7125}\right| \times 100 = 102.2534\%$$

The maximum error after the first iteration is:

209.1295%

Another iteration is needed!

Example: Unbalanced three phase load

Starting with the values obtained in iteration #1

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 172.8579 \\ -105.6067 \\ -67.0392 \\ -89.4990 \\ -62.5483 \\ 176.7125 \end{bmatrix}$$

Substituting the values from iteration #1 into the first equation

$$I_{ar} = \frac{120.00 - (-0.4516)I_{ai} - 0.0100 I_{br} - (-0.0080) I_{bi} - 0.0100 I_{cr} - (-0.0080) I_{ci}}{0.7460}$$

$$I_{ar} = 99.6000$$

Example: Unbalanced three phase load

Substituting the new value of I_{ar} and the remaining values from iteration #1 into the second equation

$$I_{ai} = \frac{0.00 - 0.4516 I_{ar} - 0.0080 I_{br} - 0.0100 I_{bi} - 0.0080 I_{cr} - 0.0100 I_{ci}}{0.7460}$$

$$I_{ai} = -57.7046$$

Substituting the new values I_{ar} , I_{ai} , and the remaining values from iteration #1 into the third equation

$$I_{br} = \frac{-60.00 - 0.0100 I_{ai} - (-0.0080) I_{ar} - (-0.5205) I_{bi} - 0.0100 I_{cr} - (-0.0080) I_{ci}}{0.7787}$$

$$I_{br} = -137.9432$$

Example: Unbalanced three phase load

Substituting the new values I_{ar} , I_{ai} , I_{br} , and the remaining values from iteration #1 into the fourth equation

$$I_{bi} = \frac{-103.9 - 0.0080 I_{ar} - 0.0100 I_{ai} - 0.5205 I_{br} - 0.0080 I_{cr} - 0.0100 I_{ci}}{0.7787}$$

$$I_{bi} = -43.1322$$

Substituting the new values I_{ar} , I_{ai} , I_{br} , I_{bi} , and the remaining values from iteration #1 into the fifth equation

$$I_{cr} = \frac{-60.00 - 0.0100 I_{ar} - (-0.0080) I_{ai} - 0.0100 I_{br} - (-0.0080) I_{bi} - (-0.6040) I_{ci}}{0.8080}$$

$$I_{cr} = 57.3157$$

Example: Unbalanced three phase load

Substituting the new values I_{ar} , I_{ai} , I_{br} , I_{bi} , I_{cr} , and the remaining value from iteration #1 into the sixth equation

$$I_{ci} = \frac{103.9 - 0.0080 I_{ar} - 0.0100 I_{ai} - 0.0080 I_{br} - 0.0100 I_{bi} - 0.6040 I_{cr}}{0.8080}$$

$$I_{ci} = 87.3718$$

The solution matrix at the end of the second iteration is:

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 99.6000 \\ -57.7046 \\ -137.9432 \\ -43.1322 \\ 57.3157 \\ 87.3718 \end{bmatrix}$$

Example: Unbalanced three phase load

Calculating the absolute relative approximate errors for the second iteration

$$\left|\varepsilon_a\right|_1 = \left|\frac{99.6000 - 172.8579}{99.6000}\right| \times 100 = 21.871\%$$

$$\left|\varepsilon_a\right|_2 = \left|\frac{-57.7046 - (-105.6067)}{-57.7046}\right| \times 100 = 23.801\%$$

$$\left|\varepsilon_a\right|_3 = \left|\frac{-137.9432 - (-67.0392)}{-137.9432}\right| \times 100 = 102.083\%$$

$$\left|\varepsilon_a\right|_4 = \left|\frac{-43.1322 - (-89.4990)}{-43.1322}\right| \times 100 = 99.055\%$$

$$\left|\varepsilon_a\right|_5 = \left|\frac{57.3157 - (-62.5483)}{57.3157}\right| \times 100 = 141.409\%$$

$$\left|\varepsilon_a\right|_6 = \left|\frac{87.3718 - 176.7125}{87.3718}\right| \times 100 = 60.754\%$$

The maximum error after the second iteration is:

141.409%

More iterations are needed!

Example: Unbalanced three phase load

Repeating more iterations, the following values are obtained

Iteration	I_{ar}	I_{ai}	I_{br}	I_{bi}	I_{cr}	I_{ci}
1	172.86	-105.61	-67.039	-89.499	-62.548	176.71
2	99.600	-57.705	-137.94	-43.132	57.316	87.372
3	127.48	-75.729	6623.1	-4562.5	-183.42	222.62
4	-18.452	2.7888	-3124.7	1953.9	150.43	23.041
5	223.616	-129.66	1222.8	-953.27	-85.658	191.70
6	58.953	-35.105	-714.26	342.26	80.196	71.328

Iteration	$\left\|\in_a\right\|_1 \%$	$\left\|\in_a\right\|_2 \%$	$\left\|\in_a\right\|_3 \%$	$\left\|\in_a\right\|_4 \%$	$\left\|\in_a\right\|_5 \%$	$\left\|\in_a\right\|_6 \%$
1	73.552	83.013	51.401	107.50	209.13	102.25
2	21.871	23.801	102.08	99.055	141.41	60.754
3	790.88	2815.5	311.96	333.51	192.01	866.22
4	108.25	102.15	355.53	304.97	275.62	87.981
5	279.31	269.37	271.20	378.52	206.81	168.77
6	61.353	61.052	583.23	246.03	387.77	52.595

Example: Unbalanced three phase load

After six iterations,

the solution matrix is

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 58.953 \\ -35.105 \\ -714.26 \\ 342.26 \\ 80.196 \\ 71.328 \end{bmatrix}$$

The maximum error after the sixth iteration is:

583.2319%

The absolute relative approximate error is still extremely high. But allowing for more iterations, the error quickly begins to converge to zero.

What could have been differently to allow for a faster convergence?

Example: Unbalanced three phase load

Repeating more iterations, the following values are obtained

Iteration	I_{ar}	I_{ai}	I_{br}	I_{bi}	I_{cr}	I_{ci}
32	120.06	-70.805	-117.52	-56.880	13.942	119.72
33	120.06	-70.805	-117.52	-56.880	13.942	119.72

Iteration	$\lvert\in_a\rvert_1 \%$	$\lvert\in_a\rvert_2 \%$	$\lvert\in_a\rvert_3 \%$	$\lvert\in_a\rvert_4 \%$	$\lvert\in_a\rvert_5 \%$	$\lvert\in_a\rvert_6 \%$
32	2.120×10^{-6}	9.636×10^{-7}	3.396×10^{-6}	4.629×10^{-6}	3.153×10^{-4}	2.747×10^{-5}
33	1.197×10^{-6}	5.519×10^{-7}	1.959×10^{-6}	2.671×10^{-6}	1.759×10^{-4}	1.532×10^{-5}

Example: Unbalanced three phase load

After 33 iterations, the solution matrix is

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 120.06 \\ -70.805 \\ -117.52 \\ -56.880 \\ 13.942 \\ 119.72 \end{bmatrix}$$

The maximum absolute relative approximate error is $1.759 \times 10^{-4}\%$

Gauss-Seidel Method: Potential Pitfall

Diagonally dominant: The coefficient on the diagonal must be at least equal to the sum of the other coefficients in that row and at least one row with a diagonal coefficient greater than the sum of the other coefficients in that row.

Non-Diagonally dominant

$$[A] = \begin{bmatrix} 2 & 5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix}$$

Diagonally dominant

$$[B] = \begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 34 & 129 \end{bmatrix}$$

Most physical systems, such as the previous example, result in simultaneous linear equations that have diagonally dominant coefficient matrices

Gauss-Seidel Method

Summary

-Advantages of the Gauss-Seidel Method

-Algorithm for the Gauss-Seidel Method

-Pitfalls of the Gauss-Seidel Method

Gauss-Seidel Method

Questions?