

Interpolation



Topic: Newton's Divided
Difference Polynomial Method

Major: Electrical



What is Interpolation ?

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, find the value of 'y' at a value of 'x' that is not given.





Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

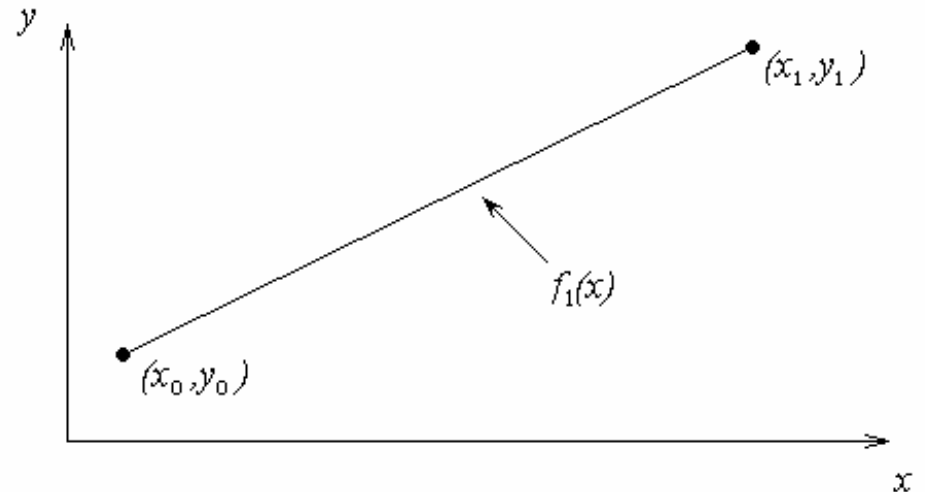
Linear interpolation: Given (x_0, y_0) , (x_1, y_1) , pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$

where

$$b_0 = f(x_0)$$

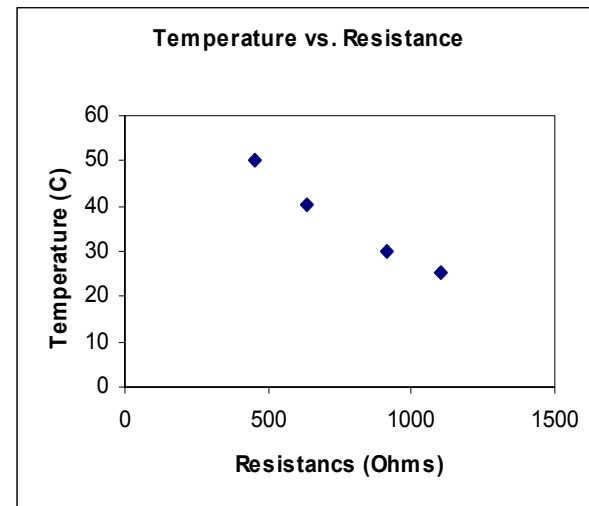
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



Example

A manufacturer of thermistors makes the following observations on a thermistor. Find the temperature corresponding to 754.8 ohms using the Newton Divided Difference method for linear interpolation.

R	T
Ohm	°C
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128



Linear Interpolation

$$T(R) = b_0 + b_1(R - R_0)$$

$$R_0 = 911.3, T(R_0) = 30.131$$

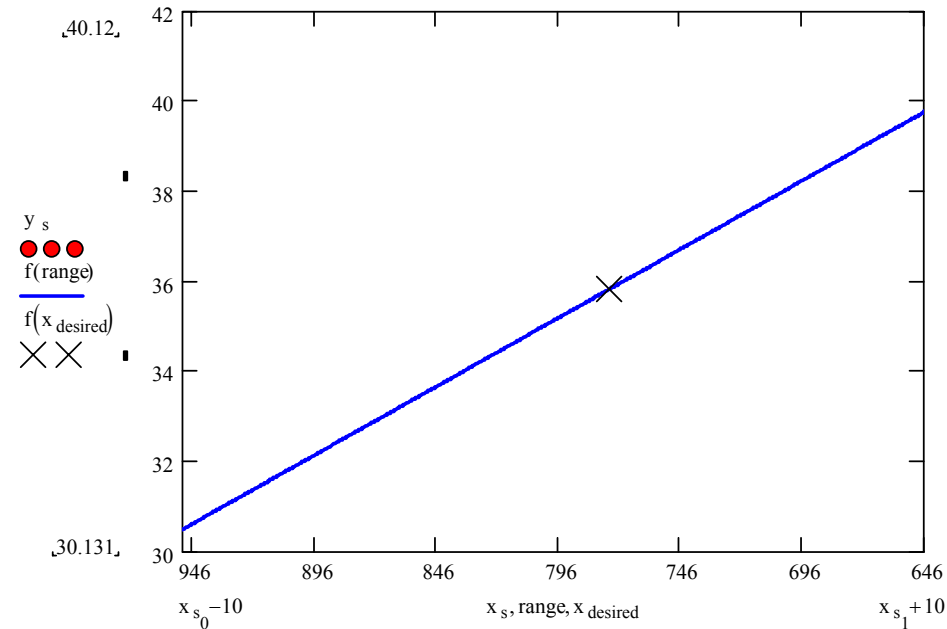
$$R_1 = 636.0, T(R_1) = 40.120$$

$$b_0 = T(R_0)$$

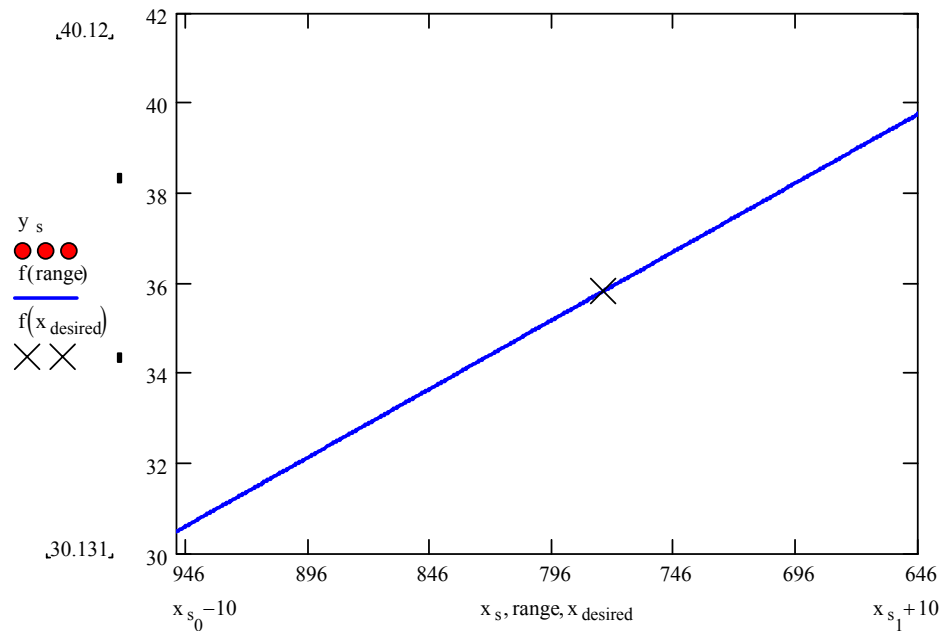
$$= 30.131$$

$$b_1 = \frac{T(R_1) - T(R_0)}{R_1 - R_0} = \frac{40.120 - 30.131}{636.0 - 911.3}$$

$$= -0.036284$$



Linear Interpolation (contd)



$$T(R) = b_0 + b_1(R - R_0)$$

$$= 30.131 - 0.036284(R - 911.3), \quad 636.0 \leq R \leq 911.3$$

At $R = 754.8$

$$T(754.8) = 30.131 - 0.036284(754.8 - 911.3)$$

$$= 35.809^\circ\text{C}$$



Quadratic Interpolation

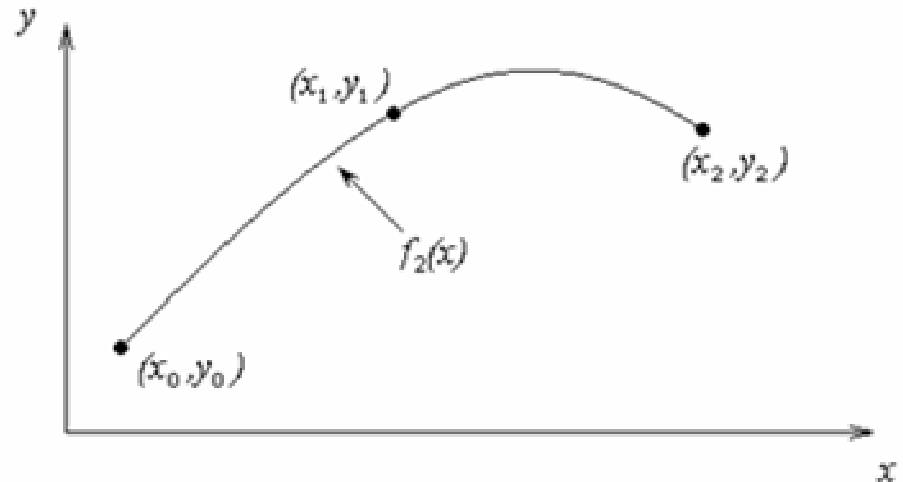
Given (x_0, y_0) , (x_1, y_1) , and (x_2, y_2) , fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

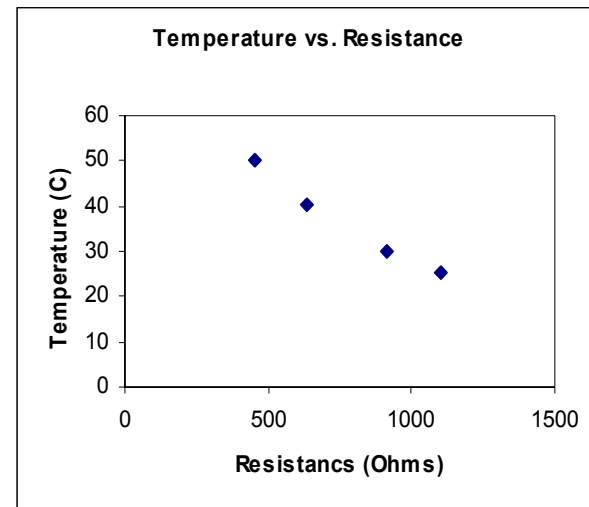
$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



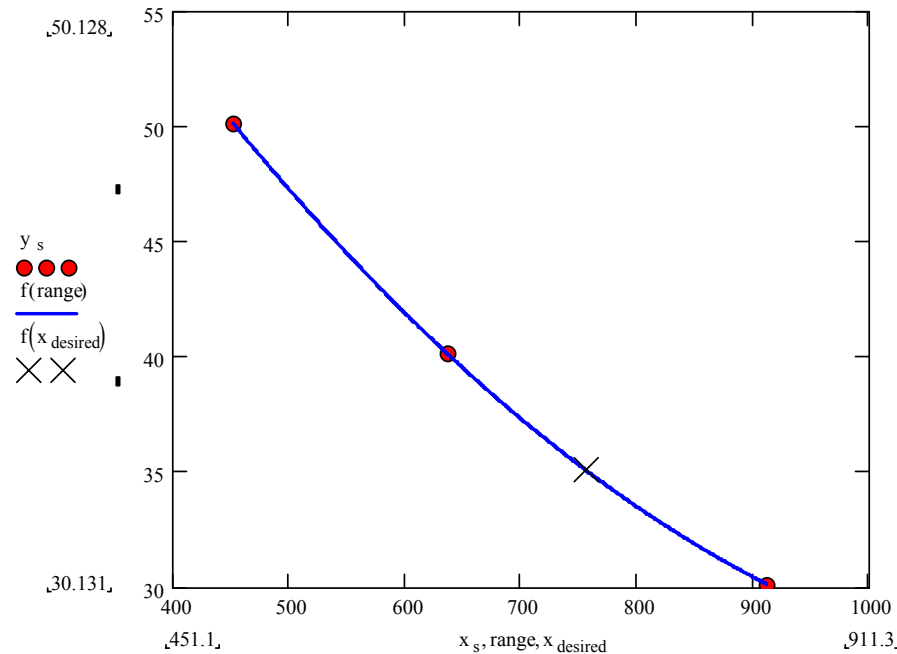
Example

A manufacturer of thermistors makes the following observations on a thermistor. Find the temperature corresponding to 754.8 ohms using the Newton Divided Difference method for quadratic interpolation.

R	T
Ohm	°C
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128



Quadratic Interpolation (contd)



$$T(R) = b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1)$$

$$R_0 = 911.3, T(R_0) = 30.131$$

$$R_1 = 636.0, T(R_1) = 40.120$$

$$R_2 = 451.1, T(R_2) = 50.128$$



Quadratic Interpolation (contd)

$$b_0 = T(R_0)$$

$$= 30.131$$

$$b_1 = \frac{T(R_1) - T(R_0)}{R_1 - R_0} = \frac{40.120 - 30.131}{636.0 - 911.3}$$

$$= -0.036284$$

$$b_2 = \frac{\frac{T(R_2) - T(R_1)}{R_2 - R_1} - \frac{T(R_1) - T(R_0)}{R_1 - R_0}}{R_2 - R_0} = \frac{\frac{50.128 - 40.120}{451.1 - 636.0} - \frac{40.120 - 30.131}{636.0 - 911.3}}{451.1 - 911.3}$$

$$= \frac{-0.054127 + 0.036284}{-460.2}$$

$$= 3.8771 * 10^{-5}$$



Quadratic Interpolation (contd)

$$\begin{aligned}T(R) &= b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1) \\ &= 30.131 - 0.036284(R - 911.3) + 3.8771 * 10^{-5}(R - 911.3)(R - 636.0), 451.1 \leq R \leq 911.3\end{aligned}$$

At $R = 754.8$,

$$\begin{aligned}T(754.8) &= 30.131 - 0.036284(754.8 - 911.3) + 3.8771 * 10^{-5}(754.8 - 911.3)(754.8 - 636.0) \\ &= 35.089^\circ C\end{aligned}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the first and second order polynomial is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{35.089 - 35.809}{35.089} \right| \times 100 \\ &= 2.0544\%\end{aligned}$$



General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$



General Form

Given $(n + 1)$ data points, $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$,

$$f_n(x) = b_0 + b_1(x - x_0) + \dots + b_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

\vdots

$$b_{n-1} = f[x_{n-1}, x_{n-2}, \dots, x_0]$$

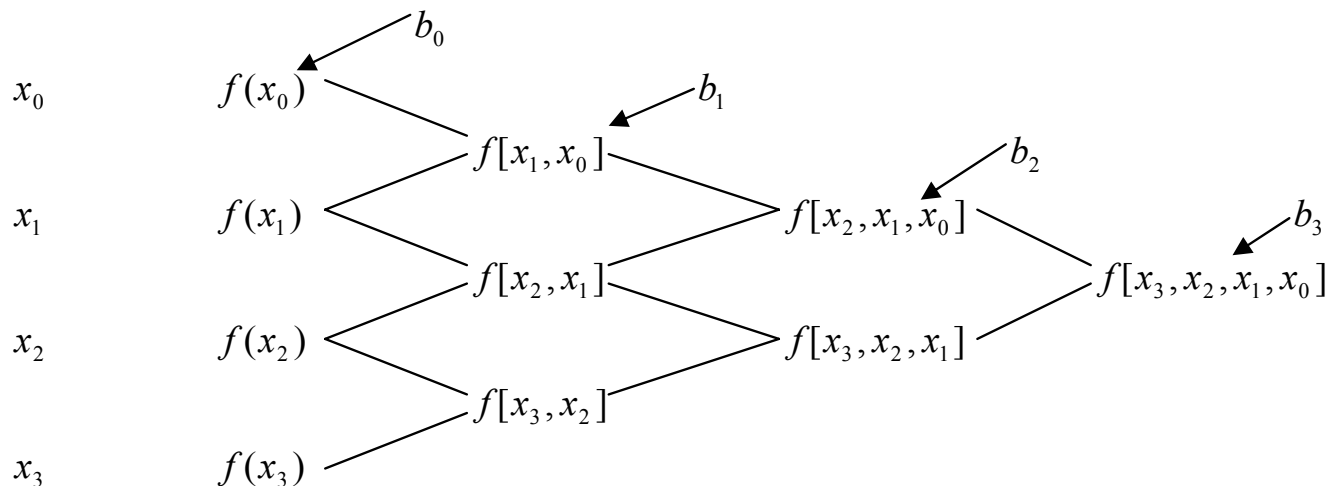
$$b_n = f[x_n, x_{n-1}, \dots, x_0]$$



General form

The third order polynomial, given (x_0, y_0) , (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) , is

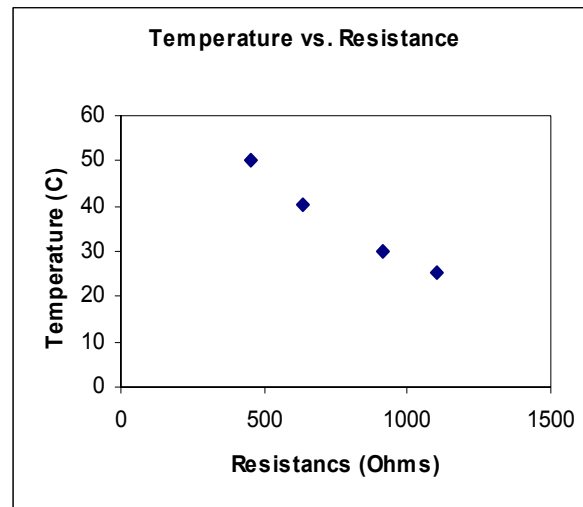
$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



Example

A manufacturer of thermistors makes the following observations on a thermistor. Find the temperature corresponding to 754.8 ohms using the Newton Divided Difference method for a third order polynomial.

R	T
Ohm	°C
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128





Example

For the third order polynomial, we choose the temperature given by

$$T(R) = b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1) + b_3(R - R_0)(R - R_1)(R - R_2)$$

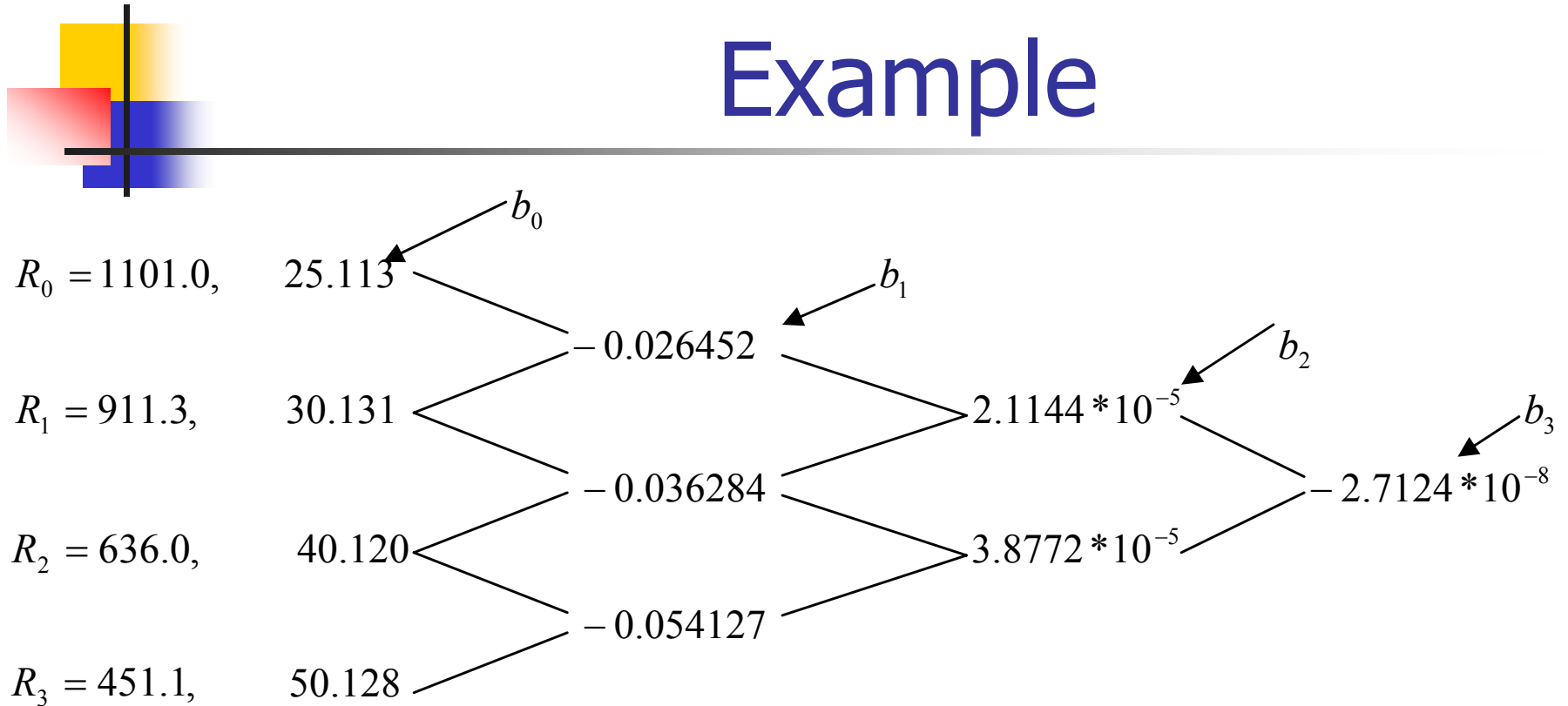
$$R_0 = 1101.0, \quad T(R_0) = 25.113$$

$$R_1 = 911.3, \quad T(R_1) = 30.131$$

$$R_2 = 636.0, \quad T(R_2) = 40.120$$

$$R_3 = 451.1, \quad T(R_3) = 50.128$$

Example



The values of the constants are found as:

$$b_0 = 25.113 \quad b_1 = -0.026452 \quad b_2 = 2.1144 * 10^{-5} \quad b_3 = -2.7124 * 10^{-8}$$



Example

$$\begin{aligned}T(R) &= b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1) + b_3(R - R_0)(R - R_1)(R - R_2) \\ &= 25.113 - 0.026452(R - 1101.0) + 2.1144 * 10^{-5}(R - 1101.0)(R - 911.3) \\ &\quad - 2.7124 * 10^{-8}(R - 1101.0)(R - 911.3)(R - 636.0)\end{aligned}$$

At $R = 754.8$,

$$\begin{aligned}T(754.8) &= 25.113 - 0.026452(754.8 - 1101.0) + 2.1144 * 10^{-5}(754.8 - 1101.0)(754.8 - 911.3) \\ &\quad - 2.7124 * 10^{-8}(754.8 - 1101.0)(754.8 - 911.3)(754.8 - 636.0) \\ &= 35.242^\circ\text{C}\end{aligned}$$

The absolute percentage relative approximate error, $|\epsilon_a|$ for the value obtained for $T(754.8)$ between second and third order polynomial is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{35.242 - 35.089}{35.242} \right| \times 100 \\ &= 0.43458\%\end{aligned}$$



Comparison Table

Order of Polynomial	1	2	3
Temperature $^{\circ}C$	35.809	35.089	35.242
Absolute Relative Approximate Error	-----	2.0544%	0.43458%



Actual Calibration

The actual calibration curve used by industry is given by

$$\frac{1}{T} = b_0 + b_1(\ln R - \ln R_0) + b_2(\ln R - \ln R_0)(\ln R - \ln R_1) + b_3(\ln R - \ln R_0)(\ln R - \ln R_1)(\ln R - \ln R_2)$$

substituting $y = \frac{1}{T}$, and $x = \ln R$, the calibration curve is given by

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

Find the calibration curve and find the temperature corresponding to 754.8 ohms. What is the difference between the results from cubic interpolation? In which method is the difference larger, if the actual measured value at 754.8 ohms is 35.285°C ?

R	T	x	y
Ohm	$^\circ\text{C}$	$\ln R$	$\frac{1}{T}$
1101.0	25.113	7.0040	0.039820
911.3	30.131	6.8149	0.033188
636.0	40.120	6.4552	0.024925
451.1	50.128	6.1117	0.019949



Actual Calibration

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

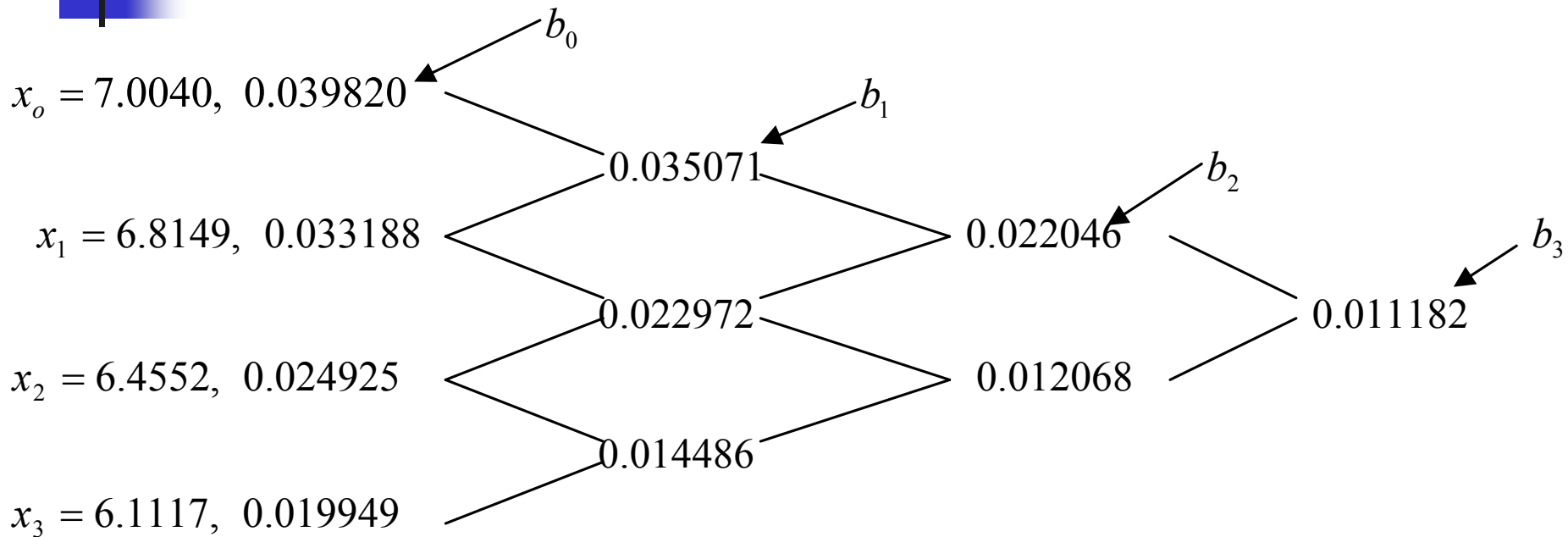
$$x_0 = 7.0040, \quad y(x_0) = 0.039820$$

$$x_1 = 6.8149, \quad y(x_1) = 0.033188$$

$$x_2 = 6.4552, \quad y(x_2) = 0.024925$$

$$x_3 = 6.1117, \quad y(x_3) = 0.019949$$

Actual Calibration



The values of the constants are found as:

$$b_0 = 0.039820$$

$$b_1 = 0.035071$$

$$b_2 = 0.022046$$

$$b_3 = 0.011182$$



Actual Calibration

$$\begin{aligned}y(x) &= b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) \\ &= 0.039820 + 0.035071(x - 7.0040) + 0.022972(x - 7.0040)(x - 6.8149) \\ &\quad + 0.011182(x - 7.0040)(x - 6.8149)(x - 6.4552), \quad 6.1117 \leq x \leq 7.0040\end{aligned}$$

Since $x = \ln 754.8 = 6.6265$

At $x = 6.6265$,

$$\begin{aligned}y(6.6265) &= 0.039820 + 0.035071(6.6265 - 7.0040) + 0.022972(6.6265 - 7.0040)(6.6265 - 6.8149) \\ &\quad + 0.011182(6.6265 - 7.0040)(6.6265 - 6.8149)(6.6265 - 6.4552) \\ &= 0.028350\end{aligned}$$

$$\begin{aligned}T = \frac{1}{y} &= \frac{1}{0.028350} \\ &= 35.274^\circ\text{C}\end{aligned}$$



Actual Calibration

Since the actual measured value at 754.8 ohms is 35.285°C , the absolute relative true error between the value used for Cubic Interpolation is

$$|\epsilon_t| = \left| \frac{35.285 - 35.242}{35.285} \right| \times 100$$
$$= 0.12186\%$$

and for actual calibration is

$$|\epsilon_t| = \left| \frac{35.285 - 35.274}{35.285} \right| \times 100$$
$$= 0.03117\%$$

Therefore, the calibration curve obtained more accurate results than a cubic polynomial interpolant given by Newton's Divided Difference method