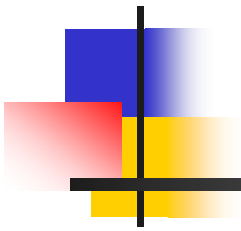


Integration



Topic: Romberg Rule

Major: Electrical Engineering

Basis of Romberg Rule

Integration

The process of measuring the area under a curve.

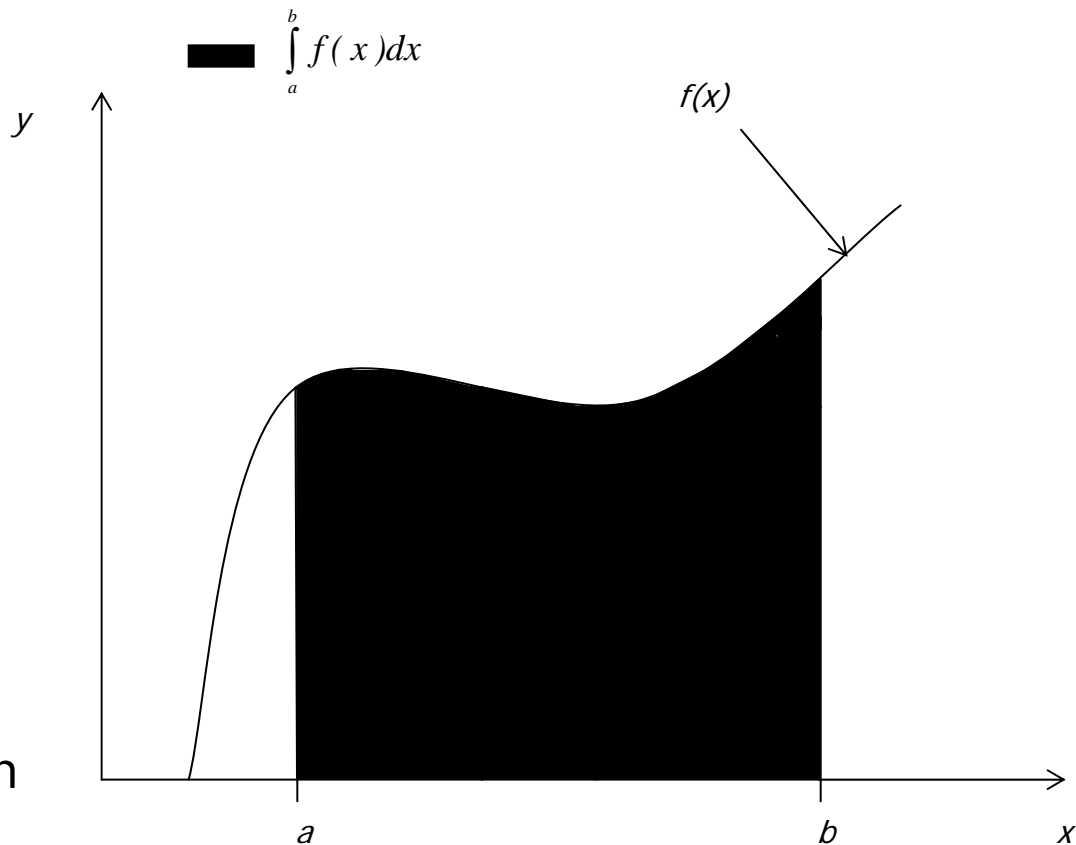
$$I = \int_a^b f(x) dx$$

Where:

$f(x)$ is the integrand

a = lower limit of integration

b = upper limit of integration





What is The Romberg Rule?

Romberg Integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integral by reducing the True Error.



Error in Multiple Segment Trapezoidal Rule

The true error in a multiple segment Trapezoidal Rule with n segments for an integral

$$I = \int_a^b f(x) dx$$

Is given by

$$E_t = \frac{(b-a)^3}{12n^2} \sum_{i=1}^n \frac{f''(\xi_i)}{n}$$

where for each i , ξ_i is a point somewhere in the domain, $[a + (i-1)h, a + ih]$.



Error in Multiple Segment Trapezoidal Rule

The term $\frac{\sum_{i=1}^n f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$.

This leads us to say that the true error, E_t previously defined can be approximated as

$$E_t \cong \alpha \frac{1}{n^2}$$

Error in Multiple Segment Trapezoidal Rule

Table 1 shows the results obtained for the integral using multiple segment Trapezoidal rule for

$$(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

n	Value	E_t	$ \epsilon_t \%$	$ \epsilon_a \%$
1	0.11489	0.86746	88.3	---
2	0.99638	-0.0140245	1.427	88.46
3	0.96092	0.021427	2.181	3.689
4	0.96968	0.012669	1.289	0.9031
5	0.974023	0.0083332	0.848	0.445
6	0.97648	0.005868	0.597	0.252
7	0.97801	0.0043459	0.442	0.155
8	0.97901	0.0033441	0.3404	0.1023

Table 1: Multiple Segment Trapezoidal Rule Values



Error in Multiple Segment Trapezoidal Rule

The true error gets approximately quartered as the number of segments is doubled. This information is used to get a better approximation of the integral, and is the basis of Richardson's extrapolation.



Richardson's Extrapolation for Trapezoidal Rule

The true error, E_t in the n -segment Trapezoidal rule is estimated as

$$E_t \cong \frac{C}{n^2}$$

where C is an *approximate constant* of proportionality. Since

$$E_t = TV - I_n$$

Where TV = true value and I_n = approx. value



Richardson's Extrapolation for Trapezoidal Rule

From the previous development, it can be shown that

$$\frac{C}{(2n)^2} \cong TV - I_{2n}$$

when the segment size is doubled and that

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

which is Richardson's Extrapolation.



Example 1

The probability for an oscillator to have its frequency within 5% of the target of 1kHz is determined by finding total area under the normal distribution function for the range in question:

$$(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

- a) Use Richardson's rule to find the distance covered. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.
- b) Find the true error, E_t for part (a).
- c) Find the absolute relative true error, $|\varepsilon_a|$ for part (a).



Solution

a) $I_2 = 0.99637$ $I_4 = 0.96969$

Using Richardson's extrapolation formula
for Trapezoidal rule

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3} \quad \text{and choosing } n=2,$$

$$\begin{aligned} TV &\cong I_4 + \frac{I_4 - I_2}{3} = 0.96969 + \frac{0.96969 - 0.99637}{3} \\ &= 0.960796 \end{aligned}$$



Solution (cont.)

- b) Since the exact value of the above integral cannot be found, we take numerical integration value using maple as exact value

$$(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
$$= 0.98236$$

Hence

$$E_t = \text{True Value} - \text{Approximate Value}$$
$$= 0.98236 - 0.960796$$
$$= -0.021564$$



Solution (cont.)

c) The absolute relative true error $|\epsilon_t|$ would then be

$$|\epsilon_t| = \left| \frac{0.98236 - 0.960796}{0.98236} \right| \times 100$$
$$= 2.19\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.



Solution (cont.)

Table 2: The values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$|\epsilon_i|$

$$(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

n	Trapezoidal Rule	for Trapezoidal Rule	Richardson's Extrapolation	for Richardson's Extrapolation
1	0.11489	88.3	--	--
2	0.99637	1.427	1.29021	31.338
4	0.96969	1.289	0.96079	2.19
8	0.97901	0.3404	0.98212	0.024

Table 2: Richardson's Extrapolation Values



Romberg Integration

Romberg integration is same as Richardson's extrapolation formula as given previously. However, Romberg used a recursive algorithm for the extrapolation. Recall

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

This can alternately be written as

$$(I_{2n})_R = I_{2n} + \frac{I_{2n} - I_n}{3} = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$



Romberg Integration

Note that the variable TV is replaced by $(I_{2n})_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign \cong is replaced by $=$ sign. Hence the estimate of the true value now is

$$TV \cong (I_{2n})_R + Ch^4$$

Where Ch^4 is an approximation of the true error.



Romberg Integration

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

It follows from the two previous expressions that the true value TV can be written as

$$\begin{aligned} TV &\cong (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15} \\ &= I_{4n} + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1} \end{aligned}$$



Romberg Integration

A general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index k represents the order of extrapolation. $k=1$ represents the values obtained from the regular Trapezoidal rule, $k=2$ represents values obtained using the true estimate as $O(h^2)$. The index j represents the more and less accurate estimate of the integral.



Example 2

The probability for an oscillator to have its frequency within 5% of the target of 1kHz is determined by finding total area under the normal distribution function for the range in question:

$$(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Use Romberg's rule to find the distance covered. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given in the Table 1.



Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = 0.11489$$

$$I_{1,2} = 0.99637$$

$$I_{1,3} = 0.96969$$

$$I_{1,4} = 0.97901$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively.



Solution (cont.)

To get the first order extrapolation values,

$$\begin{aligned}I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \\ &= 0.99637 + \frac{0.99637 - 0.11489}{3} \\ &= 1.29019\end{aligned}$$

Similarly,

$$\begin{aligned}I_{2,2} &= I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} \\ &= 0.96969 + \frac{0.96969 - 0.99637}{3} \\ &= 0.96079\end{aligned}$$

$$\begin{aligned}I_{2,3} &= I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3} \\ &= 0.97901 + \frac{0.97901 - 0.96969}{3} \\ &= 0.98211\end{aligned}$$



Solution (cont.)

For the second order extrapolation values,

$$\begin{aligned}I_{3,1} &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \\ &= 0.96079 + \frac{0.96079 - 1.29019}{15} \\ &= 0.93883\end{aligned}$$

Similarly,

$$\begin{aligned}I_{3,2} &= I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15} \\ &= 0.98211 + \frac{0.98211 - 0.96079}{15} \\ &= 0.98353\end{aligned}$$

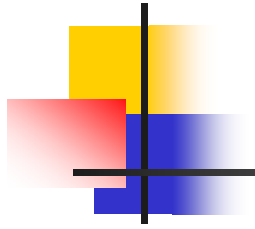


Solution (cont.)

For the third order extrapolation values,

$$\begin{aligned} I_{4,1} &= I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63} \\ &= 0.98353 + \frac{0.98353 - 0.93883}{63} \\ &= 0.98424 \end{aligned}$$

Table 3 shows these increased correct values in a tree graph.



Solution (cont.)

Table 3: Improved estimates of the integral value using Romberg Integration

		1 st Order	2 nd Order	3 rd Order
1-segment	0.11489	1.29019	0.93883	0.98424
2-segment	0.99637			
4-segment	0.96969	0.98353		
8-segment	0.97901	0.98211		