Ordinary Differential Equations

Topic: Euler Method

Major: Electrical Engineering

Authors: Autar Kaw, Charlie Barker

Euler's Method

$$\frac{dy}{dx} = f(x,y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

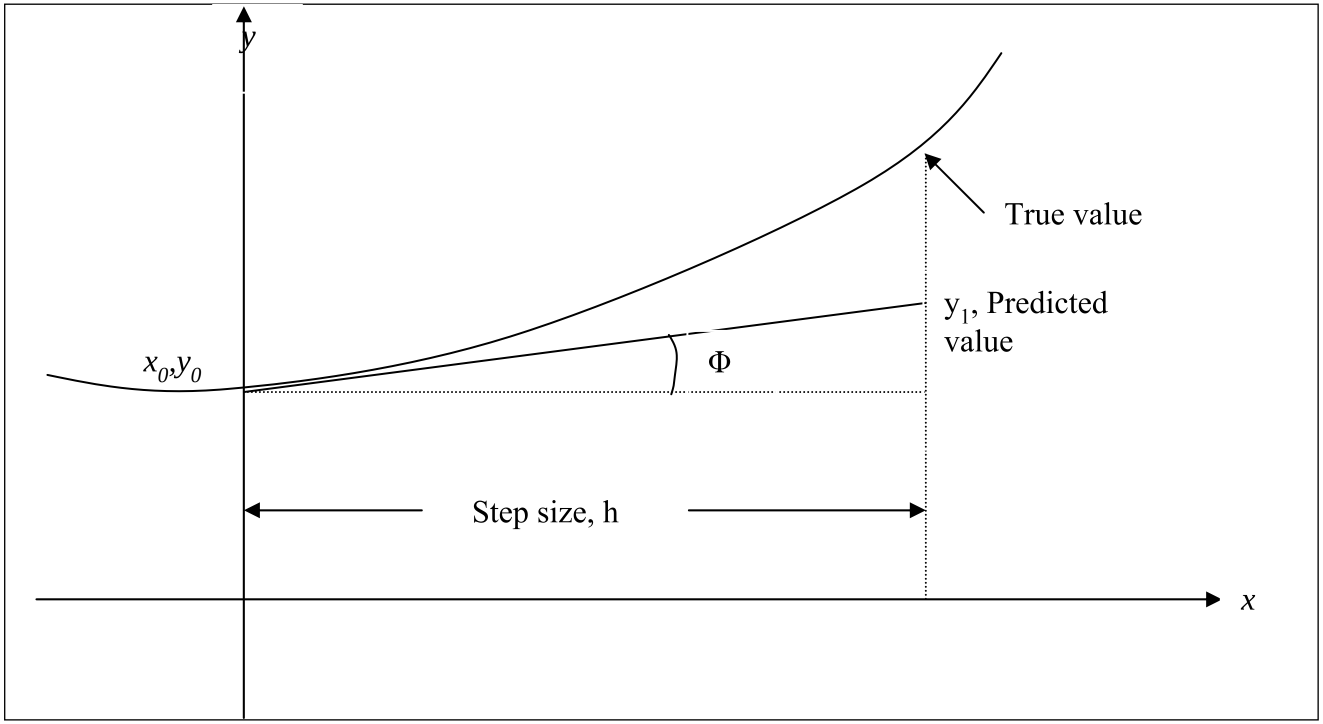

Figure 1. Graphical interpretation of the first step of Euler's method

Euler's Method

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$

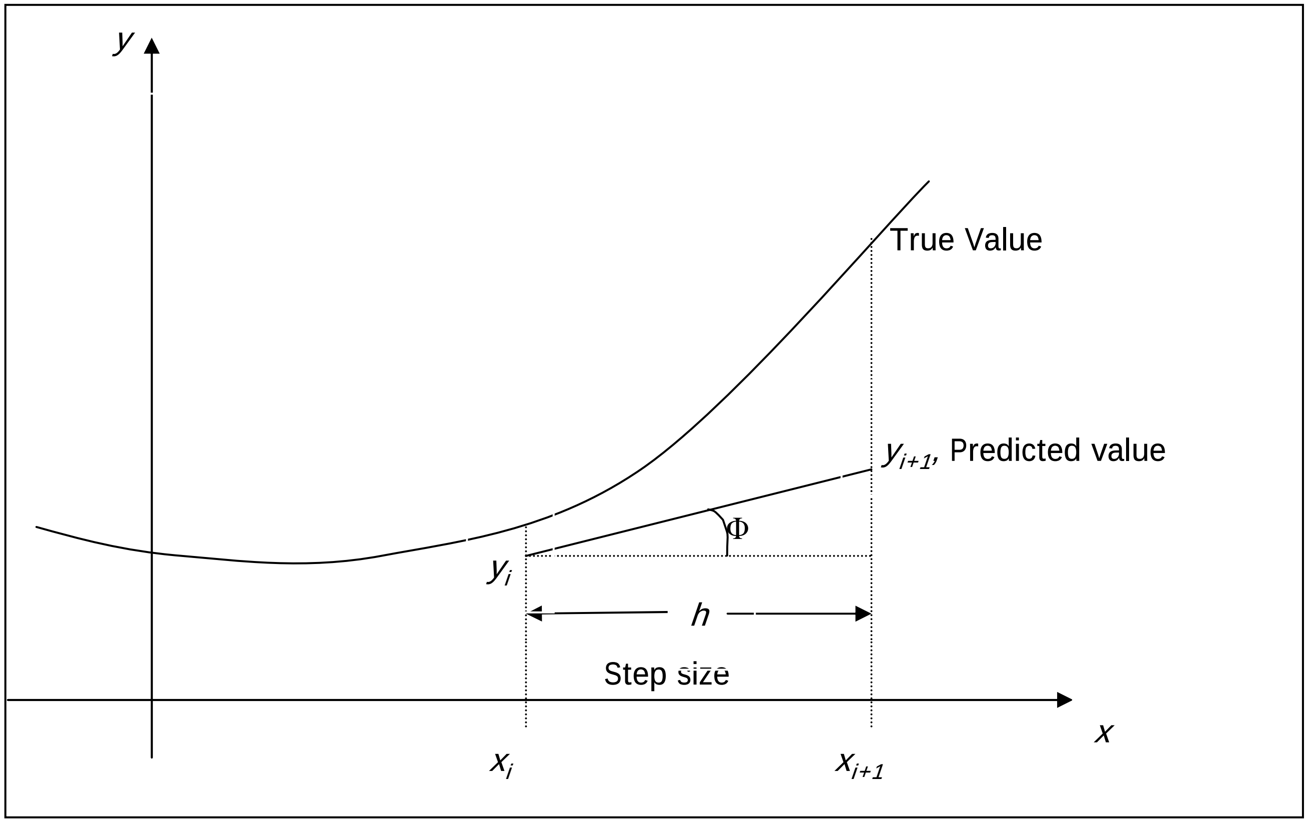

Figure 2. General graphical interpretation of Euler's method

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of 150 µF, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt} = \frac{1}{150\times 10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v(t)}{0.04},0\right)\right\} \qquad v(0)=0$$

Find voltage across the capacitor at t= 0.00004s. Use step size h=0.00002

$$\frac{dv}{dt} = \frac{1}{150\times 10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v}{0.04},0\right)\right\}$$

$$f(t,v) = \frac{1}{150\times 10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v}{0.04},0\right)\right\}$$

$$v_{i+1} = v_i + f(t_i, v_i)h$$

Solution

Step 1:

$$v_1 = v_0 + f(t_0, v_0)h$$

$$= 0 + f(0,0)0.00002$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0))| - 2 - (0)}{0.04}, 0\right)\right\}0.00002$$

$$= 0 + \left(2.6667 \times 10^{6}\right)0.00002$$

$$= 53.322V$$

v_1 is the approximate voltage at

$$t = t_1 = t_0 + h = 0 + 0.00002 = 0.00002$$

$$v_1 = v(0.00002) \cong 53.322V$$

Solution Cont

Step 2:

$$v_2 = v_1 + f(t_1, v_1)h$$

$$= 53.322 + f(0.00002, 53.322)0.00002$$

$$= 53.322 + \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00002))| - 2 - (53.322)}{0.04}, 0\right)\right\}0.00002$$

$$= 53.322 + (-0.000015000)0.00002$$

$$= 53.322V$$

v_2 is the approximate voltage at

$$t = t_2 = t_1 + h = 0.00002 + 0.00002 = 0.00004s$$

$$v_2 = v(0.00004) \cong 53.322V$$

Solution Cont

The solution to this nonlinear equation at t=0.00004 seconds is

$$v(0.00004) = 15.974V$$

Comparison of Exact and Numerical Solutions

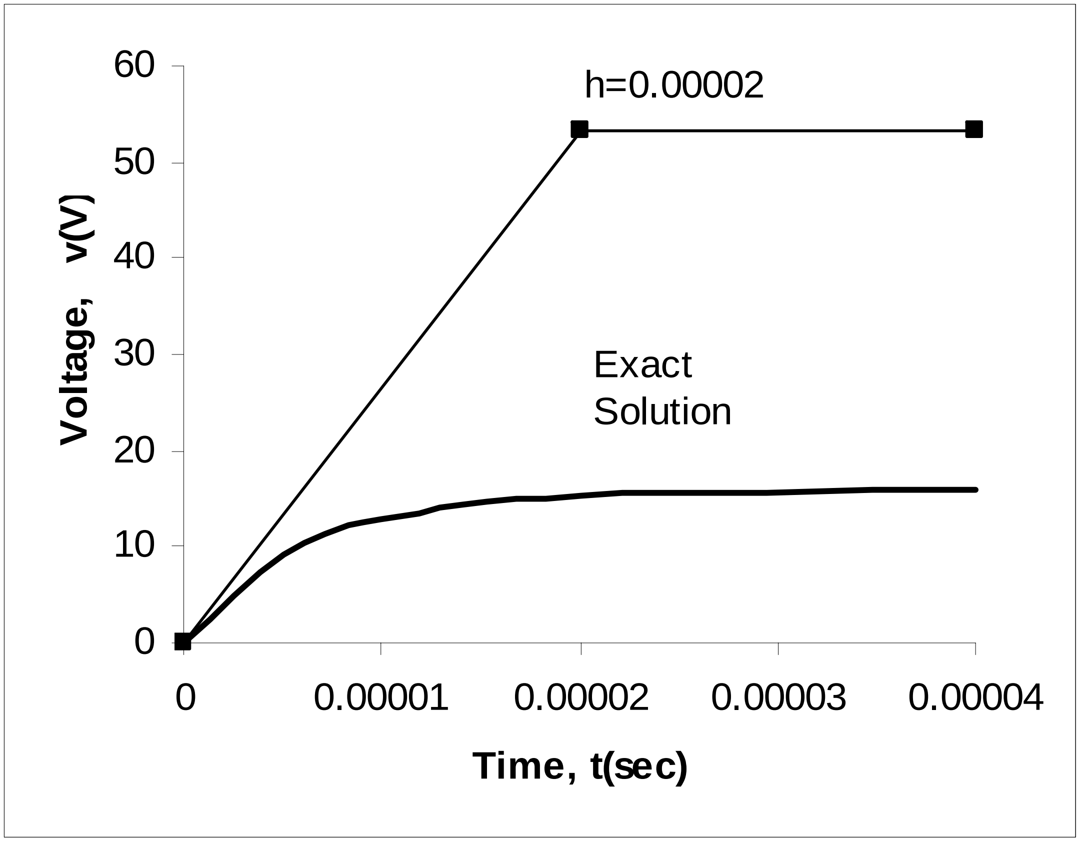

Figure 3. Comparing exact and Euler's method

Effect of step size

Table 1. Voltage at 0.00004 seconds as a function of step size, h

Step h	$v(0.00004)$	E_t	$\lvert\in_t\rvert$ %
0.00004	106.64	-90.667	567.59
0.00002	53.307	-37.333	233.71
0.00001	26.640	-10.666	66.771
0.000005	15.995	-0.021000	0.13146
0.0000025	15.992	-0.018000	0.11268

Comparison with exact results

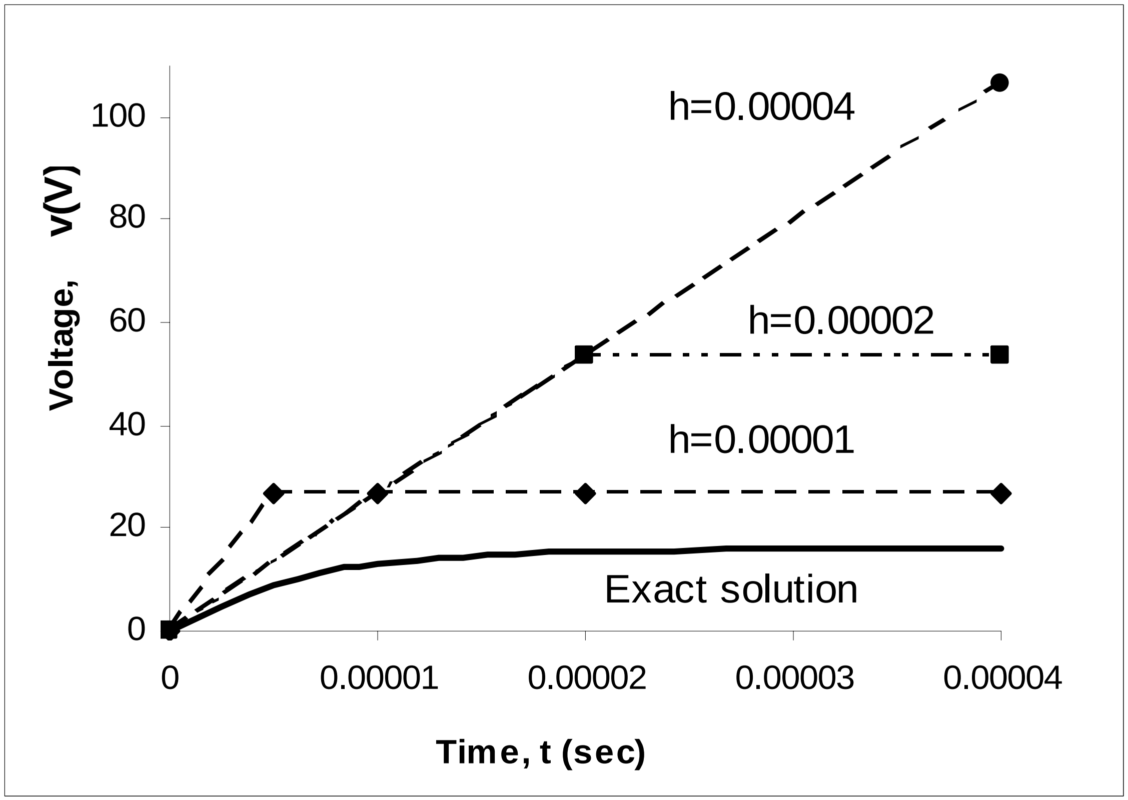

Figure 4. Comparison of Euler's method with exact solution for different step sizes

Effects of step size on Euler's Method

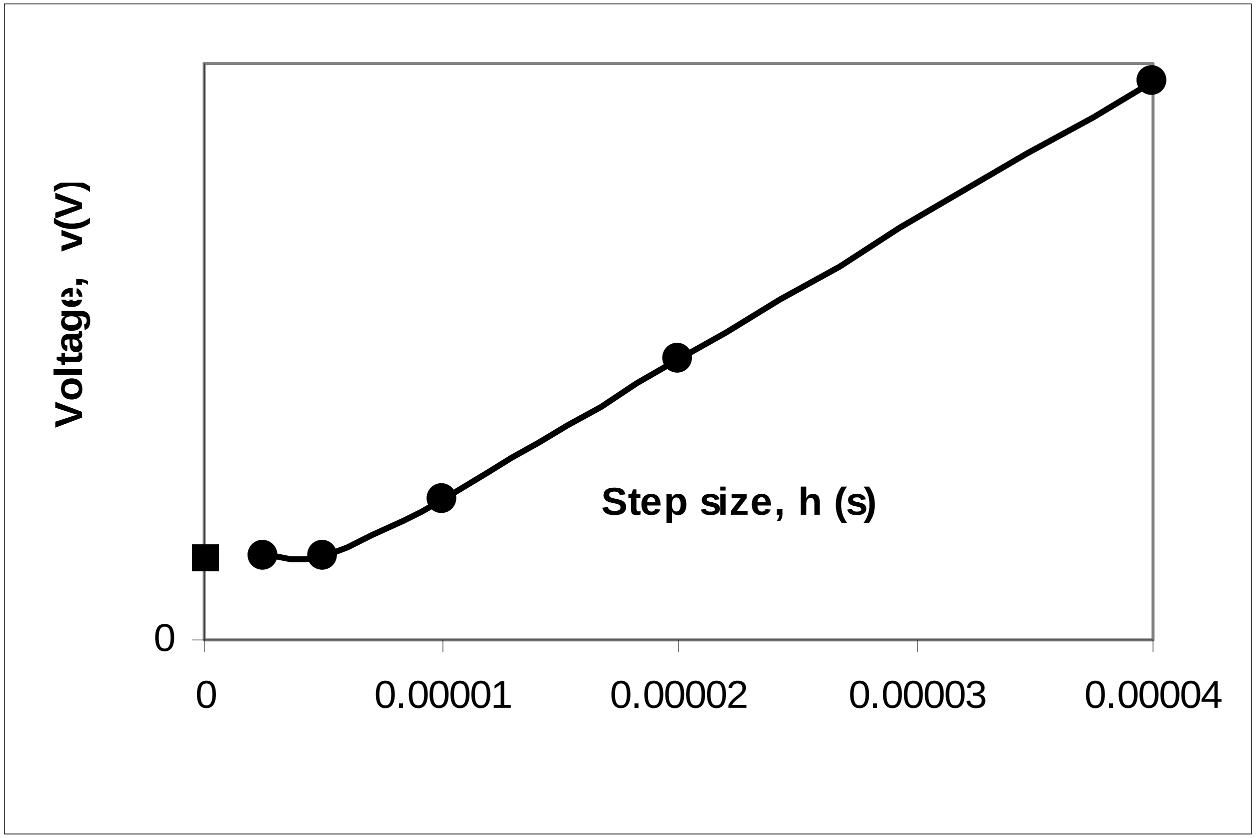

Figure 5. Effect of step size in Euler's method.

Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!} \left.\frac{d^2 y}{dx^2}\right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \left.\frac{d^3 y}{dx^3}\right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \ldots$$

$$y_{i+1} = y_i + f(x_i, y_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i, y_i)h$ are the Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \ldots \qquad E_t \propto h^2$$