Ordinary Differential Equations

Topic: Runge-Kutta 4th Order Method

Major: Electrical Engineering

Authors: Autar Kaw, Charlie Barker

Runge-Kutta 4th Order Method

For $$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f\left(x_i + h, y_i + k_3 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of 150 μF, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt} = \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(t))|-2-v(t)}{0.04},0\right)\right\} \qquad v(0)=0$$

Find voltage across the capacitor at t= 0.00004s. Use step size h=0.00002

$$\frac{dv}{dt} = \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(t))|-2-v}{0.04},0\right)\right\}$$

$$f(t,v) = \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(t))|-2-v}{0.04},0\right)\right\}$$

$$v_{i+1} = v_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

Solution

Step 1: $i = 0 \quad t_0 = 0 \quad v_0 = 0V$

$$k_1 = f(t_0, v_0) = f(0,0) = \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0))| - 2 - (0)}{0.04}, 0\right)\right\} = 2.6667\times 10^{6}$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}(0.00002), 0 + \frac{1}{2}(2.6660\times 10^{6})0.00002\right) = f(0.00001, 26.660)$$

$$= \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00001))| - 2 - (26.660)}{0.04}, 0\right)\right\} = -666.67$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}(0.00002), 0 + \frac{1}{2}(-666.67)0.00002\right) = f(0.00001, -0.0066667)$$

$$= \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00001))| - 2 - (-0.0066667)}{0.04}, 0\right)\right\} = 2.6671\times 10^{6}$$

$$k_4 = f\left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_3 h\right) = f\left(0 + \frac{1}{2}0.00002, 0 + \frac{1}{2}(2.6671\times 10^{6})0.00002\right) = f(0.00001, 26.671)$$

$$= \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00001))| - 2 - (26.671)}{0.04}, 0\right)\right\} = -666.67$$

Solution Cont

$$v_1 = v_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 0 + \frac{1}{6}\left(2.6660\times10^6 + 2(-666.67) + 2\left(2.6671\times10^6\right) + (-666.67)\right)0.00002$$

$$= 0 + \frac{1}{6}\left(7.9922\times10^6\right)0.00002$$

$$= 26.641V$$

v_1 is the approximate voltage at

$$t = t_1 = t_0 + h = 0 + 0.00002 = 0.00002$$

$$v_1 = v(0.00002) \approx 26.641V$$

Solution Cont

Step 2: $i = 1, t_1 = 0.00002, v_1 = 26.641V$

$$k_1 = f(t_1, v_1) = f(0.00002, 26.641) = \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0.00002))|-2-(26.641)}{0.04},0\right)\right\} = -666.67$$

$$k_2 = f\left(t_1+\frac{1}{2}h, v_1+\frac{1}{2}k_1h\right) = f\left(0.00002+\frac{1}{2}(0.00002), 26.641+\frac{1}{2}(-666.67)0.00002\right) = f(0.00003, 26.634)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0.00003))|-2-(26.634)}{0.04},0\right)\right\} = -666.67$$

$$k_3 = f\left(t_1+\frac{1}{2}h, v_1+\frac{1}{2}k_2h\right) = f\left(0.00002+\frac{1}{2}(0.00002), 26.641+\frac{1}{2}(-666.67)0.00002\right) = f(0.00003, 26.634)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0.00003))|-2-(26.634)}{0.04},0\right)\right\} = -666.67$$

$$k_4 = f(t_1+h, v_1+k_3h) = f\left(0.00002+\frac{1}{2}(0.00002), 26.641+\frac{1}{2}(-666.67)0.00002\right) = f(0.00003, 26.634)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0.00003))|-2-(26.634)}{0.04},0\right)\right\} = -666.67$$

Solution Cont

$$v_2 = v_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 26.641 + \frac{1}{6}\left(-666.67 + 2\left(-666.67\right) + 2\left(-666.67\right) + \left(-666.67\right)\right)0.00002$$

$$= 26.641 + \frac{1}{6}\left(-4000.0\right)0.00002$$

$$= 26.627V$$

v_2 is the approximate voltage at

$t_2 = t_1 + h$ = $0.00002 + 0.00002 = 0.00004$

$$v_2 = v\left(.00004\right) \approx 26.627V$$

Solution Cont

The **exact** solution to the differential equation at t=0.00004 seconds is

$$v(0.00004) = 15.974V$$

Comparison with exact results

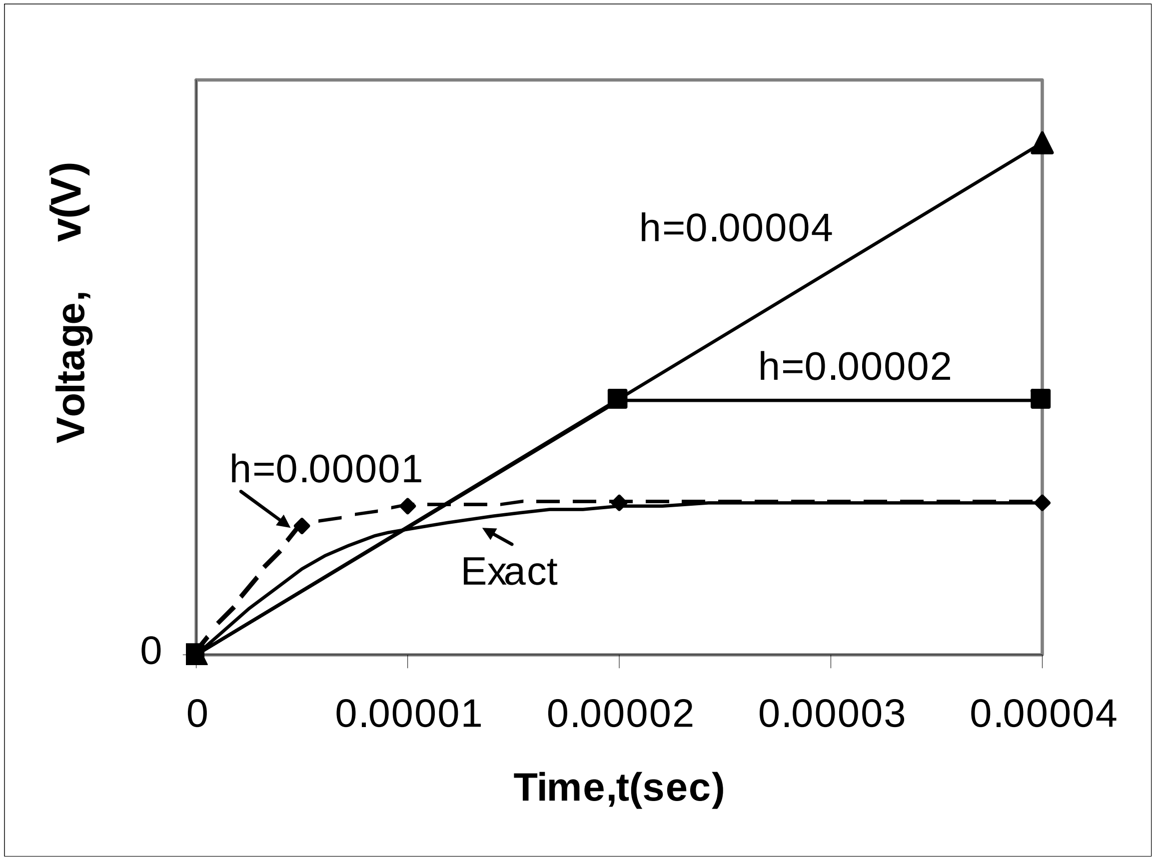

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1. Value of voltage at time, t=0.00004s for different step sizes

Step h	$v(0.00004)$	E_t	$\|\in_t\|\%$
0.00004	53.335	-37.361	233.89
0.00002	26.647	-10.673	66.815
0.00001	15.985	-0.011000	0.068862
0.000005	15.973	0.0010000	0.0062600
0.0000025	15.974	0	0

$v(0.00004) = 15.974V$ (exact)

Effects of step size on Runge-Kutta 4th Order Method

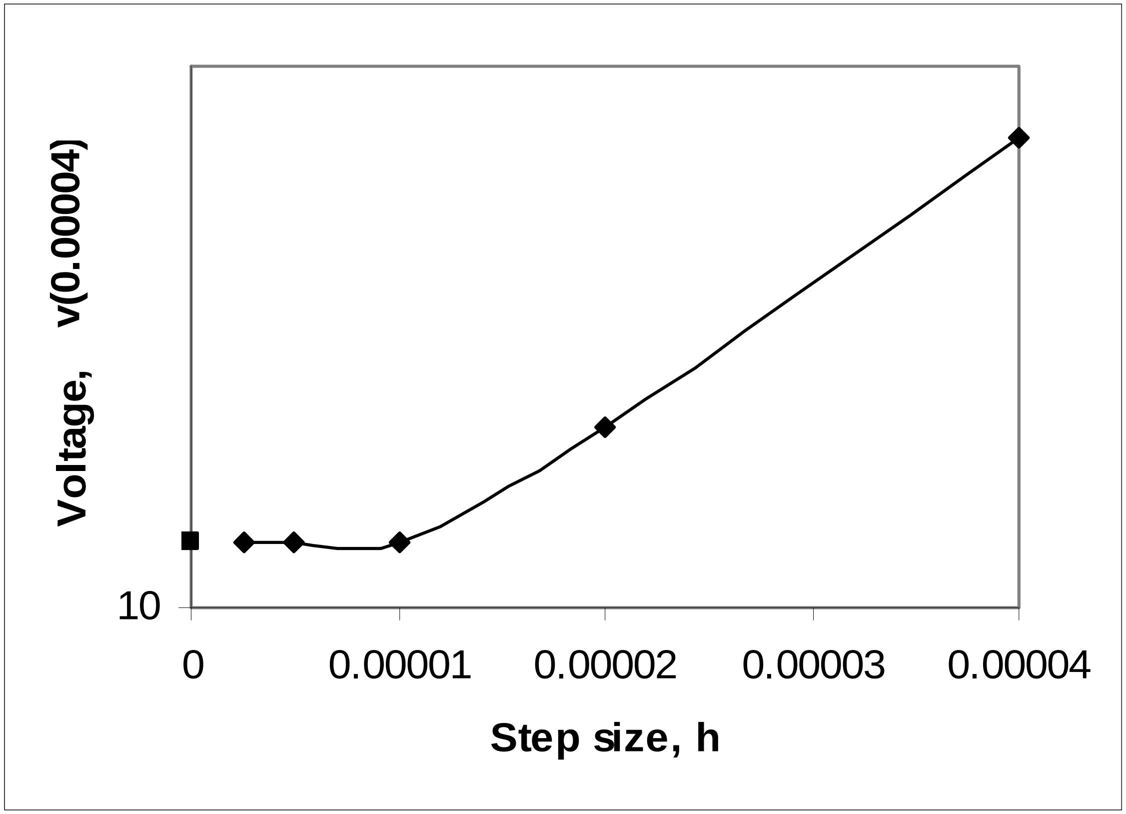

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

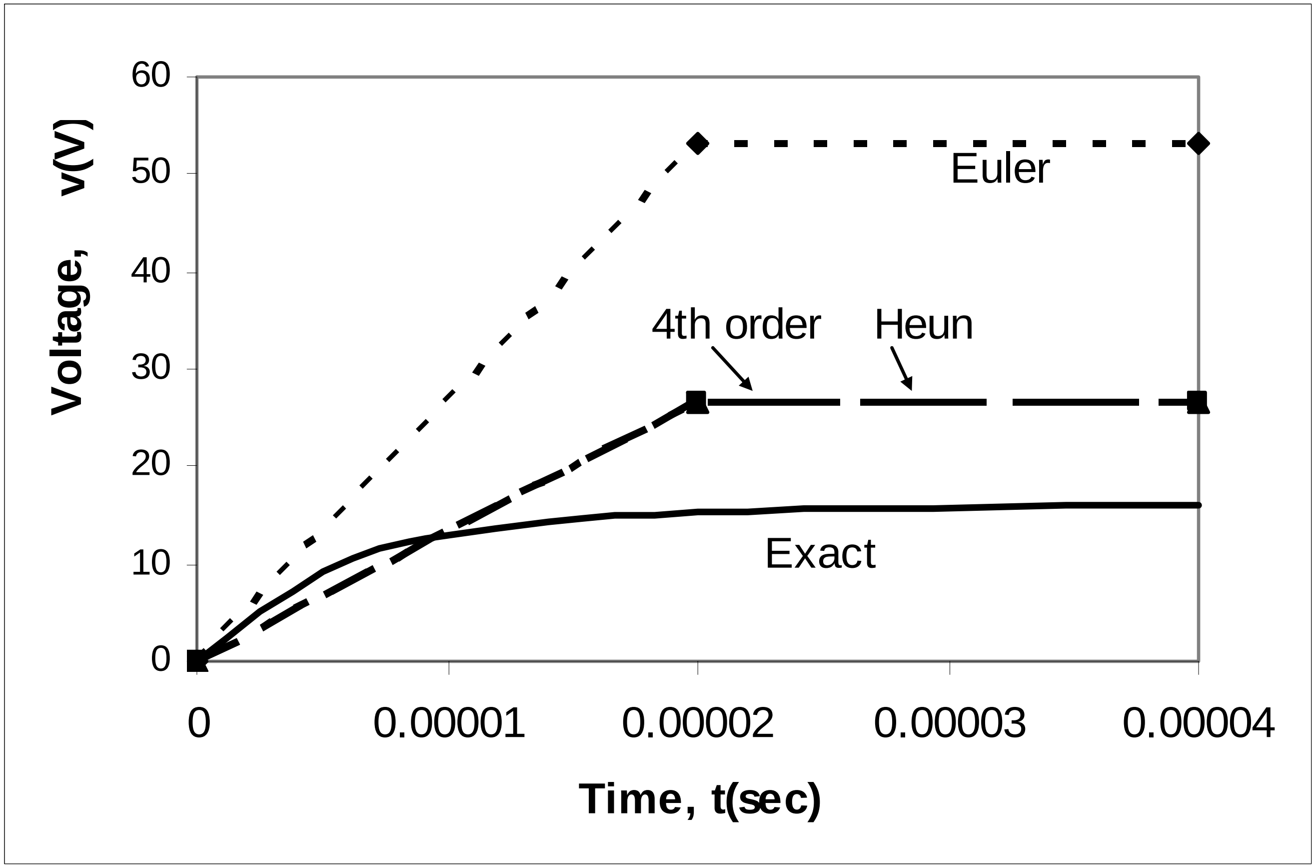

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.