05.04
Lagrangian Interpolation

After reading this chapter, you should be able to:

1. *derive Lagrangian method of interpolation,*
2. *solve problems using Lagrangian method of interpolation,and*
3. *use Lagrangian interpolants to find derivatives and integrals of discrete functions.*

What is interpolation?

Many times, a function is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). How does then one find the value of y at some other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation, but instead, is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate.

relative to other choices such as a trigonometric or exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ data points. One of the methods used to find this polynomial is called the Lagrangian method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method. We discuss the Lagrangian method in this chapter.

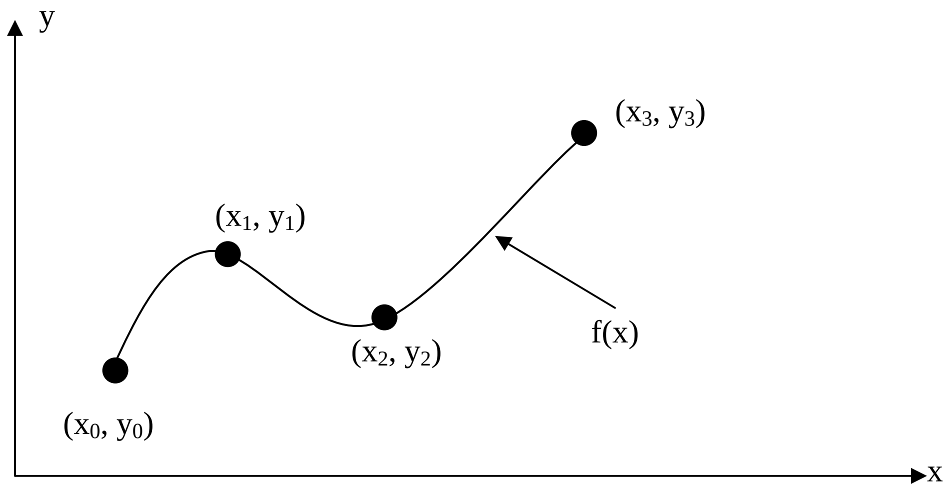

Figure 1 Interpolation of discrete data

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $n+1$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted. The application of the Lagrangian interpolation will be clarified using an example.

Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

Table 1 Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

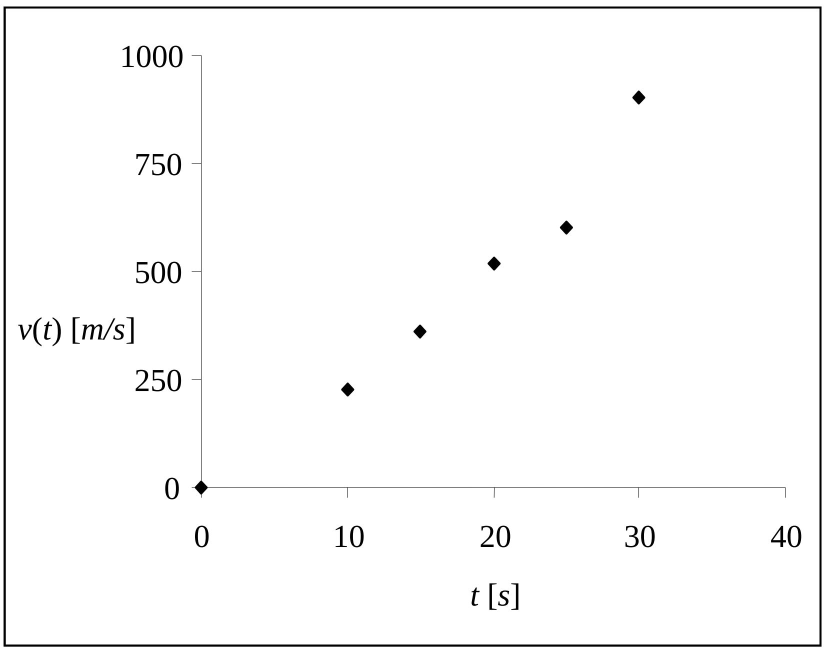

Figure 2 Velocity vs. time data for the rocket example

Determine the value of the velocity at $t = 16$ seconds using a first order Lagrange polynomial.

Solution

For the first order Lagrange polynomial interpolation (also called linear interpolation), we choose the velocity as given by

$$v(t) = \sum_{i=0}^{1} L_i(t) v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1)$$

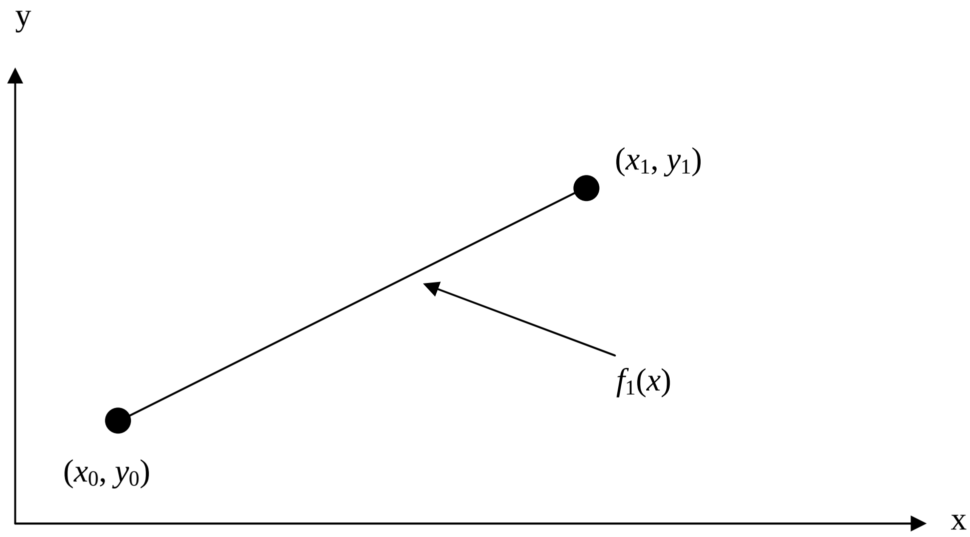

Figure 3 Linear interpolation

Since we want to find the velocity at $t = 16$ we need to choose two data points that are closest to t=16 and that also bracket $t = 16$. Those two points are $t_0 = 15$ and $t_1 = 20$.

$$t_0 = 15, v(t_0) = 362.78$$

$$t_1 = 20, v(t_1) = 517.35$$

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{1} \frac{t - t_j}{t_0 - t_j}$$

$$= \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{1} \frac{t - t_j}{t_1 - t_j}$$

$$= \frac{t - t_0}{t_1 - t_0}$$

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1)$$

$$= \frac{t - 20}{15 - 20}(362.78) + \frac{t - 15}{20 - 15}(517.35)$$

$$v(16) = \frac{16 - 20}{15 - 20}(362.78) + \frac{16 - 15}{20 - 15}(517.35)$$

$$= 0.8(362.78) + 0.2(517.35)$$
$$= 393.69 \text{ m/s}.$$

You can see that $L_0(t) = 0.8$ and $L_1(t) = 0.2$ are like weightages given to the velocities at $t = 15$ and $t = 20$ to calculate the velocity at $t = 16$.

Quadratic Interpolation

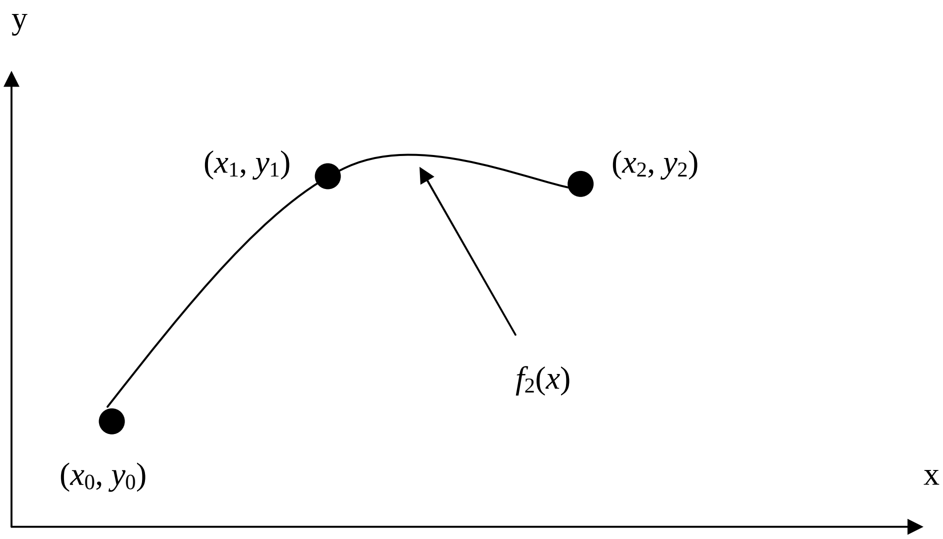

Figure 4 Quadratic interpolation

Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

Table 2 Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t=16$ seconds using second order polynomial interpolation using Lagrangian polynomial interpolation.
b) Find the absolute relative approximate error for approximation from second order polynomial.

Solution

For second order Lagrange polynomial interpolation (also called quadratic interpolation), we the velocity is given by

$$v(t)=\sum_{i=0}^{2}L_i(t)v(t_i)$$
$$=L_0(t)v(t_0)+L_1(t)v(t_1)+L_2(t)v(t_2)$$

Since we want to find the velocity at $t=16$, we need to choose data points that are closest to $t=16$ as well as bracket $t=16$. These three points are $t_0=10, t_1=15, t_2=20$.

$$t_o=10,\ v(t_0)=227.04$$
$$t_1=15,\ v(t_1)=362.78$$
$$t_2=20,\ v(t_2)=517.35$$

gives

$$L_0(t)=\prod_{\substack{j=0\\j\neq 0}}^{2}\frac{t-t_j}{t_0-t_j}$$
$$=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)$$
$$L_1(t)=\prod_{\substack{j=0\\j\neq 1}}^{2}\frac{t-t_j}{t_1-t_j}$$
$$=\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)$$
$$L_2(t)=\prod_{\substack{j=0\\j\neq 2}}^{2}\frac{t-t_j}{t_2-t_j}$$
$$=\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)$$

$$v(t)=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)v(t_0)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)v(t_1)+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)v(t_2)$$

$$v(16)=\frac{(16-15)(16-20)}{(10-15)(10-20)}(227.04)+\frac{(16-10)(16-20)}{(15-10)(15-20)}(362.78)$$
$$+\frac{(16-10)(16-15)}{(20-10)(20-15)}(517.35)$$
$$=(-0.08)(227.04)+(0.96)(362.78)+(0.12)(517.35)$$
$$=392.19\ \text{m/s}.$$

b) The absolute relative approximate error $|\in_a|$ for the second order polynomial is calculated by considering the result of the first order polynomial (Example 1) as the previous approximation

$$|\in_a| = \left|\frac{392.19 - 393.70}{392.19}\right| \times 100$$
$$= 0.38502\%$$

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds using third order Lagrangian polynomial interpolation.
b) Find the absolute relative approximate error for the third order polynomial approximation.
c) Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from $t = 11\text{s}$ to $t = 16\text{s}$.
d) Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t = 16\text{s}$.

Solution

a) For the third order Lagrangian polynomial (also called cubic interpolation), the velocity is given by

$$v(t) = \sum_{i=0}^{3} L_i(t)v(t_i)$$
$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2) + L_3(t)v(t_3)$$

Since we want to find the velocity at $t = 16$, and we are using a third order polynomial, we need to choose the four points closest to $t = 16$ and bracket $t = 16$ to evaluate it. The four points are $t_0 = 10, t_1 = 15, t_2 = 20$ and $t_3 = 22.5$.

$$t_o = 10,\ v(t_o) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$
$$t_2 = 20, \quad v(t_2) = 517.35$$
$$t_3 = 22.5, \quad v(t_3) = 602.97$$

such that

$$L_0(t) = \prod_{\substack{j=0\\ j\neq 0}}^{3} \frac{t-t_j}{t_0-t_j}$$
$$= \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)$$

$$L_1(t) = \prod_{\substack{j=0\\ j\neq 1}}^{3} \frac{t-t_j}{t_1-t_j}$$
$$= \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)$$

$$L_2(t) = \prod_{\substack{j=0\\ j\neq 2}}^{3} \frac{t-t_j}{t_2-t_j}$$
$$= \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)$$

$$L_3(t) = \prod_{\substack{j=0\\ j\neq 3}}^{3} \frac{t-t_j}{t_3-t_j}$$
$$= \left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_2}{t_3-t_2}\right)$$

$$v(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)v(t_0) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)v(t_1)$$
$$+ \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)v(t_2) + \left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_2}{t_3-t_2}\right)v(t_3)$$

$$v(16) = \frac{(16-15)(16-20)(16-22.5)}{(10-15)(10-20)(10-22.5)}(227.04) + \frac{(16-10)(16-20)(16-22.5)}{(15-10)(15-20)(15-22.5)}(362.78)$$
$$+ \frac{(16-10)(16-15)(16-22.5)}{(20-10)(20-15)(20-22.5)}(517.35)$$
$$+ \frac{(16-10)(16-15)(16-20)}{(22.5-10)(22.5-15)(22.5-20)}(602.97)$$
$$= (-0.0416)(227.04) + (0.832)(362.78) + (0.312)(517.35) + (-0.1024)(602.97)$$
$$= 392.06 \text{ m/s}.$$

b) The absolute percentage relative approximate error, $\left|\in_a\right|$ for the value obtained for $v(16)$ can be obtained by comparing the result with that obtained using the second order polynomial (Example 2)

$$\left|\in_a\right| = \left|\frac{392.06 - 392.19}{392.06}\right| \times 100$$
$$= 0.033427\%$$

c) The distance covered by the rocket between $t = 11\,\text{s}$ to $t = 16\,\text{s}$ can be calculated from the interpolating polynomial as

$$v(t) = \frac{(t-15)(t-20)(t-22.5)}{(10-15)(10-20)(10-22.5)}(227.04) + \frac{(t-10)(t-20)(t-22.5)}{(15-10)(15-20)(15-22.5)}(362.78)$$
$$+ \frac{(t-10)(t-15)(t-22.5)}{(20-10)(20-15)(20-22.5)}(517.35)$$
$$+ \frac{(t-10)(t-15)(t-20)}{(22.5-10)(22.5-15)(22.5-20)}(602.97), 10 \le t \le 22.5$$
$$= \frac{(t^2-35t+300)(t-22.5)}{(-5)(-10)(-12.5)}(227.04) + \frac{(t^2-30t+200)(t-22.5)}{(5)(-5)(-7.5)}(362.78)$$
$$+ \frac{(t^2-25t+150)(t-22.5)}{(10)(5)(-2.5)}(517.35) + \frac{(t^2-25t+150)(t-20)}{(12.5)(7.5)(2.5)}(602.97)$$
$$10 \le t \le 22.5$$
$$= (t^3 - 57.5t^2 + 1087.5t - 6750)(-0.36326) + (t^3 - 52.5t^2 + 875t - 4500)(1.9348)$$
$$+ (t^3 - 47.5t^2 + 712.5t - 3375)(-4.1388) + (t^3 - 45t^2 + 650t - 3000)(2.5727)$$
$$10 \le t \le 22.5$$
$$= -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3,\ 10 \le t \le 22.5$$

Note that the polynomial is valid between $t = 10$ and $t = 22.5$ and hence includes the limits of $t = 11$ and $t = 16$.

So

$$s(16) - s(11) = \int_{11}^{16} v(t)dt$$
$$\approx \int_{11}^{16} (-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3)dt$$

$$= \left[-4.245t + 21.265\frac{t^2}{2} + 0.13195\frac{t^3}{3} + 0.00544\frac{t^4}{4} \right]_{11}^{16}$$

$$= 1605 \text{ m}.$$

d) The acceleration at $t = 16$ is given by

$$a(16) = \frac{d}{dt}v(t)\Big|_{t=16}$$

Given that

$$v(t) = -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3,\ 10 \le t \le 22.5$$

$$a(t) = \frac{d}{dt}v(t)$$

$$= \frac{d}{dt}\left(-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3\right)$$

$$= 21.265 + 0.26390t + 0.01632t^2,\ 10 \le t \le 22.5$$

$$a(16) = 21.265 + 0.26390(16) + 0.01632(16)^2$$

$$= 29.665 \text{ m/s}^2$$

Note: There is no need to get the simplified third order polynomial expression to conduct the differentiation. An expression of the form

$$L_0(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)$$

gives the derivative without expansion as

$$\frac{d}{dt}(L_0(t)) = = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right) + \left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right) + \left(\frac{t-t_3}{t_0-t_3}\right)\left(\frac{t-t_1}{t_0-t_1}\right)$$

INTERPOLATION	
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