Chapter 05.03
Newton's Divided Difference Interpolation

After reading this chapter, you should be able to:

1. *derive Newton's divided difference method of interpolation,*
2. *apply Newton's divided difference method of interpolation,*
3. *apply Newton's divided difference method interpolants to find derivatives and integrals.*

What is interpolation?

Many a times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how does then one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called interpolation.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called extrapolation.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. One of the methods of interpolation is called the Newton's divided difference polynomial method. Other methods include the direct method and the Lagrangian interpolation method. We discuss the Newton's divided difference polynomial method in this chapter.

Newton's Divided Difference Polynomial Method

To illustrate this method, linear and quadratic interpolation is presented first. Then, the general form of Newton's divided difference polynomial method is presented. To illustrate the general form, cubic interpolation is shown.

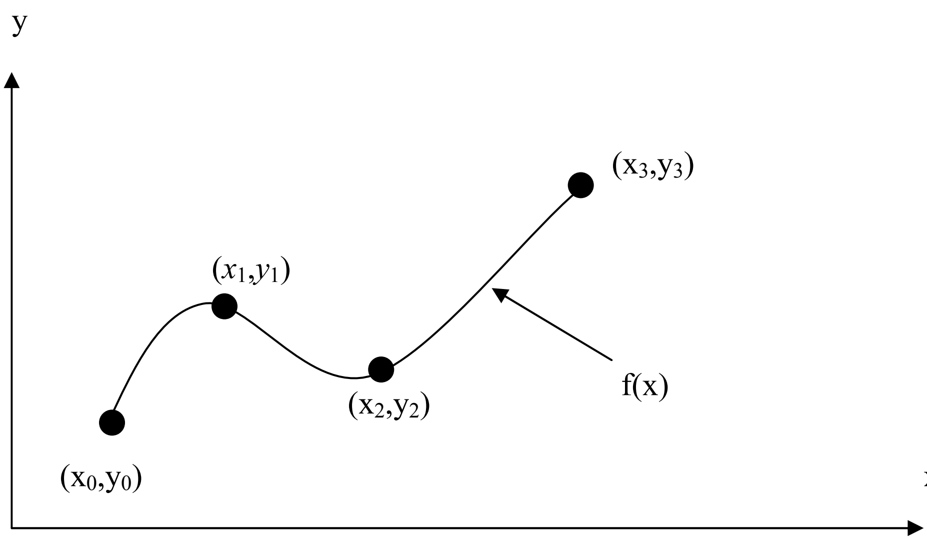

Figure 1 Interpolation of discrete data

Linear Interpolation

Given (x_0, y_0) and (x_1, y_1), fit a linear interpolant through the data. Noting $y = f(x)$ and $y_1 = f(x_1)$, assume the linear interpolant, $f_1(x)$ is given by (Figure 2)

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at $x = x_0$,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0,$$

and at $x = x_1$,

$$f_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$

$$= f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

giving the linear interpolant to be

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

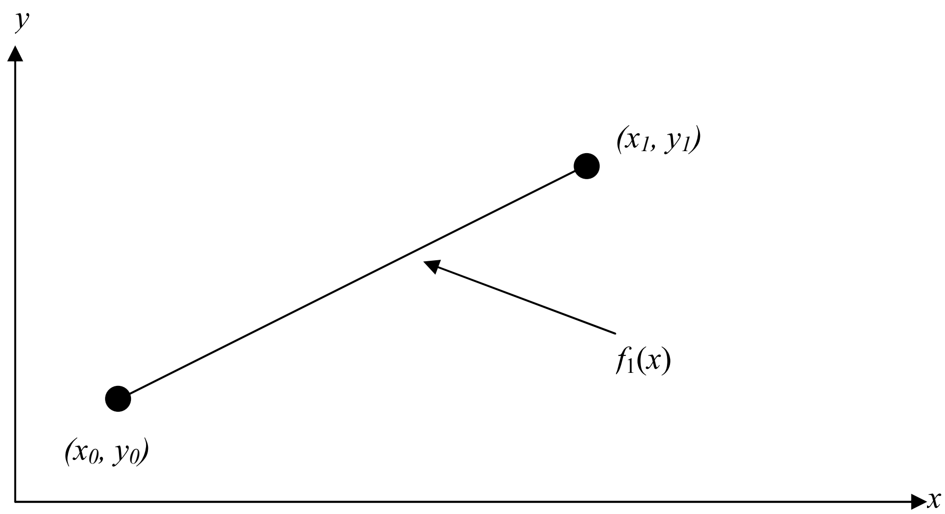

Figure 2 Linear Interpolation

Example 1

The upward velocity of a rocket is given as a function of time in Table 1 (Figure 3).

Table 1 Velocity as a function of time

$t(s)$	$v(t)(m/s)$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at $t = 16$ seconds using first order polynomial interpolation by Newton's Divided Difference Polynomial method.

Solution

For the linear interpolation, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0)$$

Since we want the velocity at $t = 16$, we need to choose two data points that are closest to $t = 16$ and that also bracket $t = 16$. Those two points are $t = 15$ and $t = 20$.

$$t_0 = 15,\ v(t_0) = 362.78$$

$$t_1 = 20,\ v(t_1) = 517.35$$

gives

$$b_0 = v(t_0)$$

$$= 362.78$$
$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0}$$
$$= \frac{517.35 - 362.78}{20 - 15}$$
$$= 30.914$$

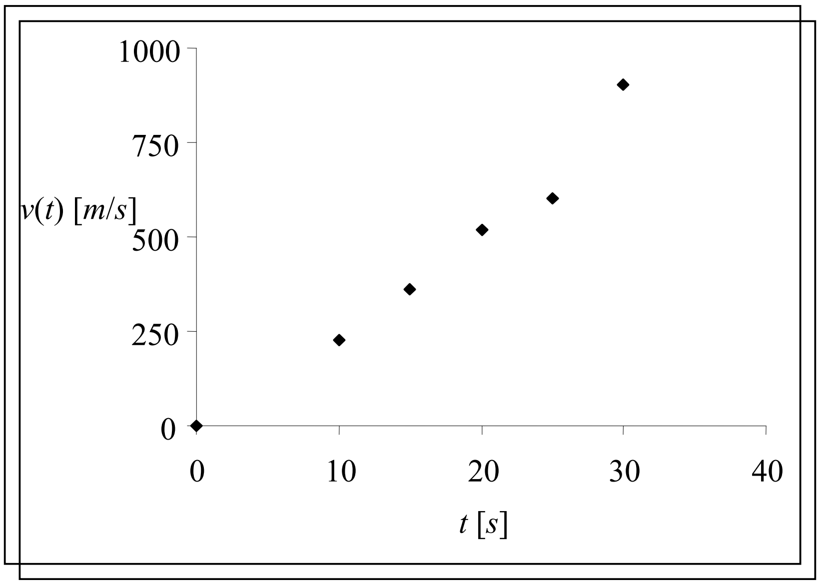

Figure 3 Graph of velocity vs. time data for the rocket example

Hence

$$v(t) = b_0 + b_1(t - t_0)$$
$$= 362.78 + 30.914(t - 15), \qquad 15 \le t \le 20$$

At $t = 16$

$$v(16) = 362.78 + 30.914(16 - 15)$$
$$= 393.69 \text{ m/s}.$$

If we expand

$$v(t) = 362.78 + 30.914(t - 15), \qquad 15 \le t \le 20$$

we get

$$v(t) = -100.93 + 30.914t, \qquad 15 \le t \le 20$$

and this is the same expression as obtained in the direct method.

Quadratic Interpolation

Given $(x_0, y_0), (x_1, y_1)$, and (x_2, y_2), fit a quadratic interpolant through the data. Noting $y = f(x)$, $y_0 = f(x_0)$, $y_1 = f(x_1)$, and $y_2 = f(x_2)$, assume the quadratic interpolant $f_2(x)$ is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At $x = x_0$

$$f_2(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1)$$

$$= b_0$$

$$b_0 = f(x_0)$$

At $x = x_1$

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$

$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

Giving

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Hence the quadratic interpolant is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1)$$

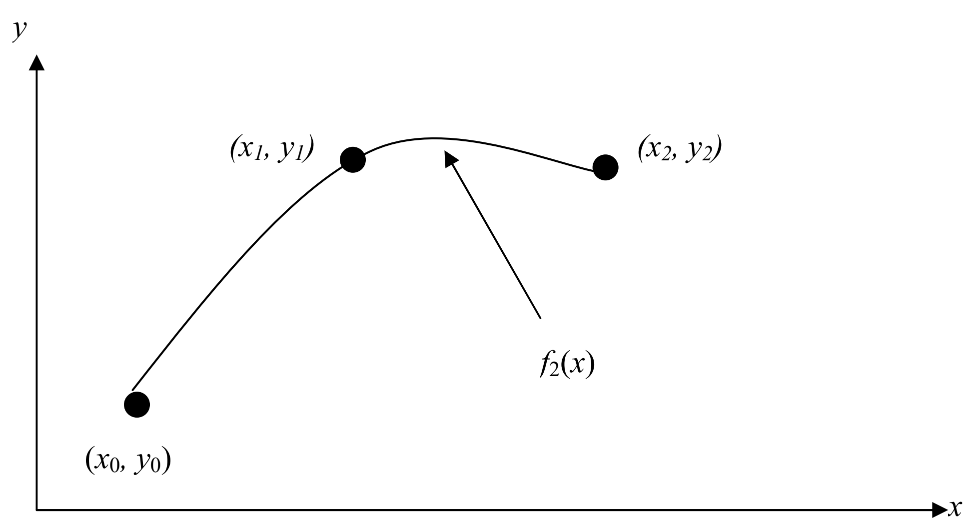

Figure 4 Quadratic interpolation

Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

Table 2 Velocity as a function of time

$t\,(s)$	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at $t=16$ seconds using second order polynomial interpolation using Newton's divided difference polynomial method.

Solution

For quadratic interpolation, the velocity is given by

$$v(t)=b_0+b_1(t-t_0)+b_2(t-t_0)(t-t_1)$$

Since we want to find the velocity at $t=16,$ we need to choose the three data points that are closest to $t=16$ and that also bracket $t=16$. These three points are $t_0=10, t_1=15,$ and $t_2=20$.

$$t_0=10,\ v(t_0)=227.04,$$
$$t_1=15,\ v(t_1)=362.78,$$
$$t_2=20,\ v(t_2)=517.35,$$

then

$$b_0=v(t_0)$$
$$=227.04$$
$$b_1=\frac{v(t_1)-v(t_0)}{t_1-t_0}$$
$$=\frac{362.78-227.04}{15-10}$$
$$=27.148$$
$$b_2=\frac{\dfrac{v(t_2)-v(t_1)}{t_2-t_1}-\dfrac{v(t_1)-v(t_0)}{t_1-t_0}}{t_2-t_0}$$
$$=\frac{\dfrac{517.35-362.78}{20-15}-\dfrac{362.78-227.04}{15-10}}{20-10}$$
$$=\frac{30.914-27.148}{10}$$
$$=0.37660$$

Then

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$
$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \qquad 10 \le t \le 20$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15)$$
$$= 392.19 \text{ m/s}$$

If we expand,

$$v(t) = 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \; 10 \le t \le 20$$

we get

$$v(t) = 12.05 + 17.733t + 0.37660t^2, \quad 10 \le t \le 20$$

This is the same expression obtained by the direct method.

General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for the Newton's Divided Difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Note that b_0, b_1, and b_2 are finite divided differences. b_0, b_1, and b_2 are the first, second, and third finite divided differences, respectively. We denote the first divided difference by

$$f[x_0] = f(x_0),$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

and the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$
$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0},$$

where $f[x_0]$, $f[x_1, x_0]$, and $f[x_2, x_1, x_0]$ are called bracketed functions of their variables enclosed in square brackets.

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for $n+1$ data points, $(x_0, y_0),(x_1, y_1),......,(x_{n-1}, y_{n-1}),(x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + ... + b_n(x - x_0)(x - x_1)...(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$
$$b_1 = f[x_1, x_0]$$
$$b_2 = f[x_2, x_1, x_0]$$
$$\vdots$$
$$b_{n-1} = f[x_{n-1}, x_{n-2}, ..., x_0]$$
$$b_n = f[x_n, x_{n-1}, ..., x_0]$$

where the definition of the m^{th} divided difference is

$$b_m = f[x_m, ..., x_0]$$
$$= \frac{f[x_m, ..., x_1] - f[x_{m-1}, ..., x_0]}{x_m - x_0}$$

From the above definition, it can be seen that the divided differences are calculated recursively.

For an example of a third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3),

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

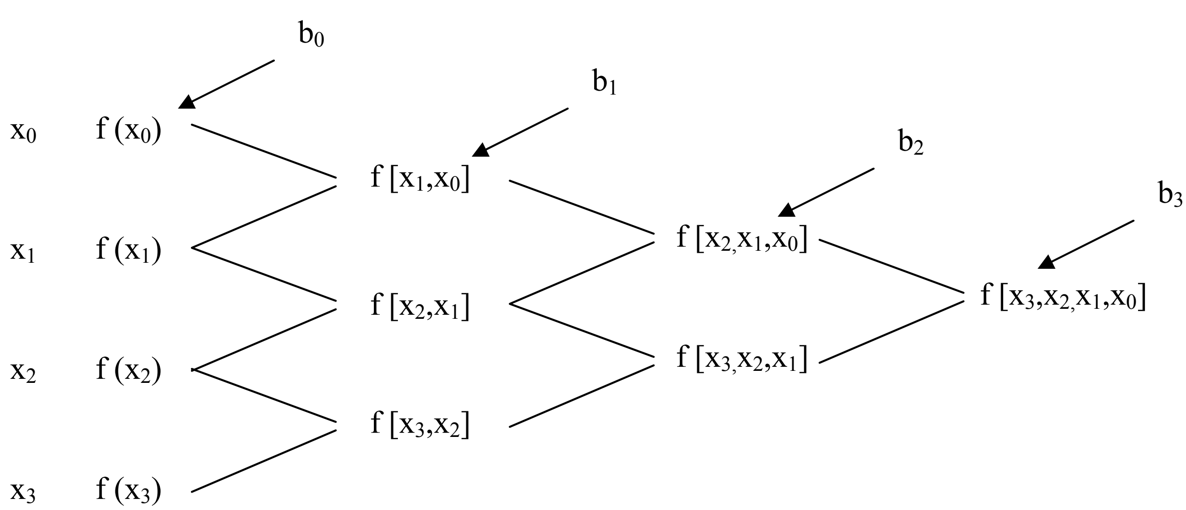

Figure 5. Table of divided differences for a cubic polynomial

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 Velocity as a function of time

t(s)	$v(t)$(m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t=16$ seconds using third order polynomial interpolation using Newton's Divided Difference polynomial.

b) Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from $t=11\text{s}$ to $t=16\text{s}$.

c) Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t=16\text{s}$.

Solution

a) For a third order polynomial, the velocity is given by

$$v(t)=b_0+b_1(t-t_0)+b_2(t-t_0)(t-t_1)+b_3(t-t_0)(t-t_1)(t-t_2)$$

Since we want to find the velocity at $t=16$, we need to choose the four data points that are closest to $t=16$ and that also bracket $t=16$. Those four data points are $t_0=10$, $t_1=15$, $t_2=20$, and $t_3=22.5$.

$$t_0=10,\quad v(t_0)=227.04,$$

$$t_1=15,\quad v(t_1)=362.78,$$

$$t_2=20,\quad v(t_2)=517.35,$$

$$t_3=22.5,\ v(t_3)=602.97,$$

then

$$b_0=v[t_0]$$
$$=v(t_0)$$
$$=227.04$$

$$b_1=v[t_1,t_0]$$
$$=\frac{v(t_1)-v(t_0)}{t_1-t_0}$$
$$=\frac{362.78-227.04}{15-10}$$
$$=27.148$$

$$b_2=v[t_2,t_1,t_0]$$
$$=\frac{v[t_2,t_1]-v[t_1,t_0]}{t_2-t_0}$$

$$v[t_2,t_1]=\frac{v(t_2)-v(t_1)}{t_2-t_1}$$

$$= \frac{517.35 - 362.78}{20 - 15}$$

$$= 30.914$$

$$v[t_1, t_0] = 27.148$$

$$b_2 = \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0}$$

$$= \frac{30.914 - 27.148}{20 - 10}$$

$$= 0.37660$$

$$b_3 = v[t_3, t_2, t_1, t_0]$$

$$= \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0}$$

$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1}$$

$$v[t_3, t_2] = \frac{v(t_3) - v(t_2)}{t_3 - t_2}$$

$$= \frac{602.97 - 517.35}{22.5 - 20}$$

$$= 34.248$$

$$v[t_2, t_1] = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

$$= \frac{517.35 - 362.78}{20 - 15}$$

$$= 30.914$$

$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1}$$

$$= \frac{34.248 - 30.914}{22.5 - 15}$$

$$= 0.44453$$

$$v[t_2, t_1, t_0] = 0.37660$$

$$b_3 = \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0}$$

$$= \frac{0.44453 - 0.37660}{22.5 - 10}$$

$$= 5.4347 \times 10^{-3}$$

Hence

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

$$= 227.04 + 27.148(t-10) + 0.37660(t-10)(t-15)$$
$$+ 5.5347\times10^{-3}(t-10)(t-15)(t-20)$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16-10) + 0.37660(16-10)(16-15)$$
$$+ 5.5347\times10^{-3}(16-10)(16-15)(16-20)$$
$$= 392.06\,\text{m/s}$$

b) The distance covered by the rocket between $t = 11\text{s}$ and $t = 16\text{s}$ can be calculated from the interpolating polynomial

$$v(t) = 227.04 + 27.148(t-10) + 0.37660(t-10)(t-15) + 5.5347\times10^{-3}(t-10)(t-15)(t-20) \qquad 10 \le t \le 22.5$$
$$= -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3 \qquad 10 \le t \le 22.5$$

Note that the polynomial is valid between t=10 and t=22.5 and hence includes the limits of $t = 11$ and $t = 16$.

So

$$s(16) - s(11) = \int_{11}^{16} v(t)dt$$
$$= \int_{11}^{16} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3)dt$$
$$= \left[-4.2541t + 21.265\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4}\right]_{11}^{16}$$
$$= 1605\,\text{m}$$

(c) The acceleration at $t = 16$ is given by

$$a(16) = \frac{d}{dt}v(t)\Big|_{t=16}$$
$$a(t) = \frac{d}{dt}v(t)$$
$$= \frac{d}{dt}\left(-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3\right)$$
$$= 21.265 + 0.26408t + 0.016304t^2$$
$$a(16) = 21.265 + 0.26408(16) + 0.016304(16)^2$$
$$= 29.664\,\text{m/s}^2$$

INTERPOLATION	
Topic	Newton's divided difference interpolation
Summary	Textbook notes on Newton's divided difference interpolation.
Major	General Engineering
Authors	Autar Kaw, Michael Keteltas
Last Revised	December 7, 2008
Web Site	http://numericalmethods.eng.usf.edu