# Integration

## Topic: Romberg Rule

Major: General Engineering

# Basis of Romberg Rule

**Integration**

The process of measuring the area under a curve.

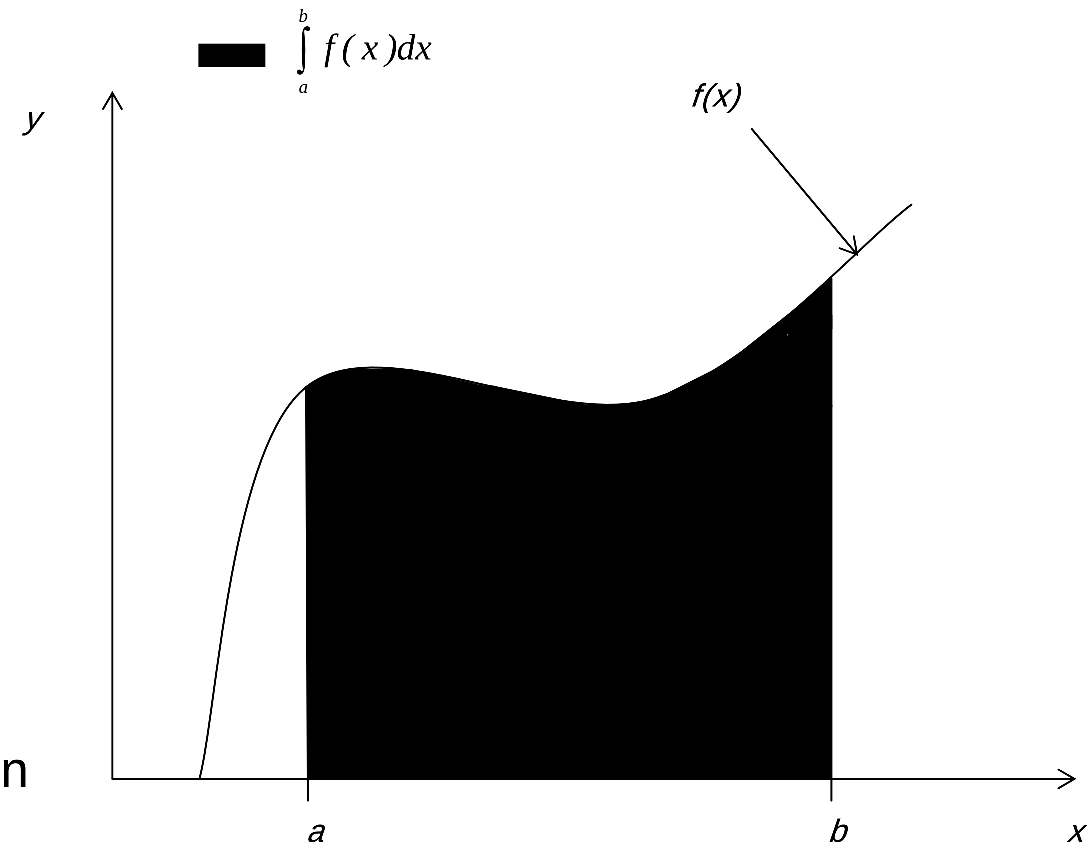
$$I = \int_a^b f(x)dx$$

Where:

*f(x)* is the integrand

a= lower limit of integration

b= upper limit of integration



# What is The Romberg Rule?

Romberg Integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integral by reducing the True Error.

# Error in Multiple Segment Trapezoidal Rule

The true error in a multiple segment Trapezoidal Rule with n segments for an integral

$$I = \int_a^b f(x)dx$$

Is given by

$$E_t = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

where for each *i*, $\xi_i$ is a point somewhere in the domain , $[a+(i-1)h, a+ih]$ .

# Error in Multiple Segment Trapezoidal Rule

The term $\dfrac{\sum_{i=1}^{n} f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$.

This leads us to say that the true error, $E_t$ previously defined can be approximated as

$$E_t \cong \alpha \frac{1}{n^2}$$

# Error in Multiple Segment Trapezoidal Rule

Table 1 shows the results obtained for the integral using multiple segment Trapezoidal rule for

$$x = \int_{8}^{30} \left( 2000\, ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

| n | Value | $E_t$ | $\lvert \in_t \rvert \%$ | $\lvert \in_a \rvert \%$ |
|---|---|---|---|---|
| 1 | 11868 | 807 | 7.296 | --- |
| 2 | 11266 | 205 | 1.854 | 5.343 |
| 3 | 11153 | 91.4 | 0.8265 | 1.019 |
| 4 | 11113 | 51.5 | 0.4655 | 0.3594 |
| 5 | 11094 | 33.0 | 0.2981 | 0.1669 |
| 6 | 11084 | 22.9 | 0.2070 | 0.09082 |
| 7 | 11078 | 16.8 | 0.1521 | 0.05482 |
| 8 | 11074 | 12.9 | 0.1165 | 0.03560 |

**Table 1: Multiple Segment Trapezoidal Rule Values**

# Error in Multiple Segment Trapezoidal Rule

The true error gets approximately quartered as the number of segments is doubled. This information is used to get a better approximation of the integral, and is the basis of Richardson's extrapolation.

# Richardson's Extrapolation for Trapezoidal Rule

The true error, $E_t$ in the *n*-segment Trapezoidal rule is estimated as

$$E_t \cong \frac{C}{n^2}$$

where *C* is an *approximate constant* of proportionality. Since

$$E_t = TV - I_n$$

Where TV = true value and $I_n$ = approx. value

# Richardson's Extrapolation for Trapezoidal Rule

From the previous development, it can be shown that

$$\frac{C}{(2n)^2} \cong TV - I_{2n}$$

when the segment size is doubled and that

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

which is Richardson's Extrapolation.

# Example 1

The vertical distance covered by a rocket from 8 to 30 seconds is given by

$$x = \int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

a) Use Richardson's rule to find the distance covered. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.
b) Find the true error, $E_t$ for part (a).
c) Find the absolute relative true error, $|\varepsilon_a|$ for part (a).

# Solution

a) $I_2 = 11266m$ $I_4 = 11113m$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$ and choosing *n=2*,

$$TV \cong I_4 + \frac{I_4 - I_2}{3} = 11113 + \frac{11113 - 11266}{3}$$

$$= 11062m$$

# Solution (cont.)

b) The exact value of the above integral is

$$x = \int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

$$= 11061\ m$$

Hence

$$E_t = True\ Value - Approximate\ Value$$

$$= 11061 - 11062$$

$$= -1\ m$$

# Solution (cont.)

c) The absolute relative true error $|\in_t|$ would then be

$$|\in_t| = \left|\frac{11061-11062}{11061}\right| \times 100$$

$$= 0.00904\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

# Solution (cont.)

Table 2: The values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$x = \int_8^{30} \left( 2000 \, ln\left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

.

| n | Trapezoidal Rule | $\left| \in_t \right|$ for Trapezoidal Rule | Richardson's Extrapolation | $\left| \in_t \right|$ for Richardson's Extrapolation |
|---|---|---|---|---|
| 1 | 11868 | 7.296 | -- | -- |
| 2 | 11266 | 1.854 | 11065 | 0.03616 |
| 4 | 11113 | 0.4655 | 11062 | 0.009041 |
| 8 | 11074 | 0.1165 | 11061 | 0.0000 |

**Table 2: Richardson's Extrapolation Values**

# Romberg Integration

Romberg integration is same as Richardson's extrapolation formula as given previously. However, Romberg used a recursive algorithm for the extrapolation. Recall

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

This can alternately be written as

$$(I_{2n})_R = I_{2n} + \frac{I_{2n} - I_n}{3} \quad = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$

# Romberg Integration

Note that the variable $TV$ is replaced by $\left(I_{2n}\right)_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign $\cong$ is replaced by = sign.

Hence the estimate of the true value now is

$$TV \cong \left(I_{2n}\right)_R + Ch^4$$

Where $Ch^4$ is an approximation of the true error.

# Romberg Integration

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

It follows from the two previous expressions that the true value TV can be written as

$$TV \cong (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15}$$

$$= I_{4n} + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1}$$

# Romberg Integration

A general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index *k* represents the order of extrapolation. k=1 represents the values obtained from the regular Trapezoidal rule, k=2 represents values obtained using the true estimate as $O(h^2)$. The index *j* represents the more and less accurate estimate of the integral.

# Example 2

The vertical distance covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$x = \int_{8}^{30} \left( 2000 \, ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Use Romberg's rule to find the distance covered. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given in the Table 1.

# Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = 11868 \qquad I_{1,2} = 11266$$

$$I_{1,3} = 11113 \qquad I_{1,4} = 11074$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively.

# Solution (cont.)

To get the first order extrapolation values,

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$

$$= 11266 + \frac{11266 - 11868}{3}$$

$$= 11065$$

Similarly,

$$I_{2,2} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3}$$

$$= 11113 + \frac{11113 - 11266}{3}$$

$$= 11062$$

$$I_{2,3} = I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3}$$

$$= 11074 + \frac{11074 - 11113}{3}$$

$$= 11061$$

# Solution (cont.)

For the second order extrapolation values,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

$$= 11062 + \frac{11062 - 11065}{15}$$

$$= 11062$$

Similarly,

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$

$$= 11061 + \frac{11061 - 11062}{15}$$

$$= 11061$$

# Solution (cont.)

For the third order extrapolation values,

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$

$$= 11061 + \frac{11061 - 11062}{63}$$

$$= 11061m$$

Table 3 shows these increased correct values in a tree graph.

# Solution (cont.)

**Table 3: Improved estimates of the integral value using Romberg Integration**

| | | *First Order* | *Second Order* | *Third Order* |
|---|---|---|---|---|
| *1-segment* | 11868 | | | |
| | | 11065 | | |
| *2-segment* | 1126 | | 11062 | |
| | | 11062 | | 11061 |
| *4-segment* | 11113 | | 11061 | |
| | | 11061 | | |
| *8-segment* | 11074 | | | |