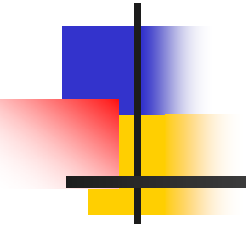




Differentiation-Discrete Functions



Major: Industrial Engineering

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Forward Difference Approximation

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite ' Δx '

$$f'(x) \cong \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Graphical Representation Of Forward Difference Approximation

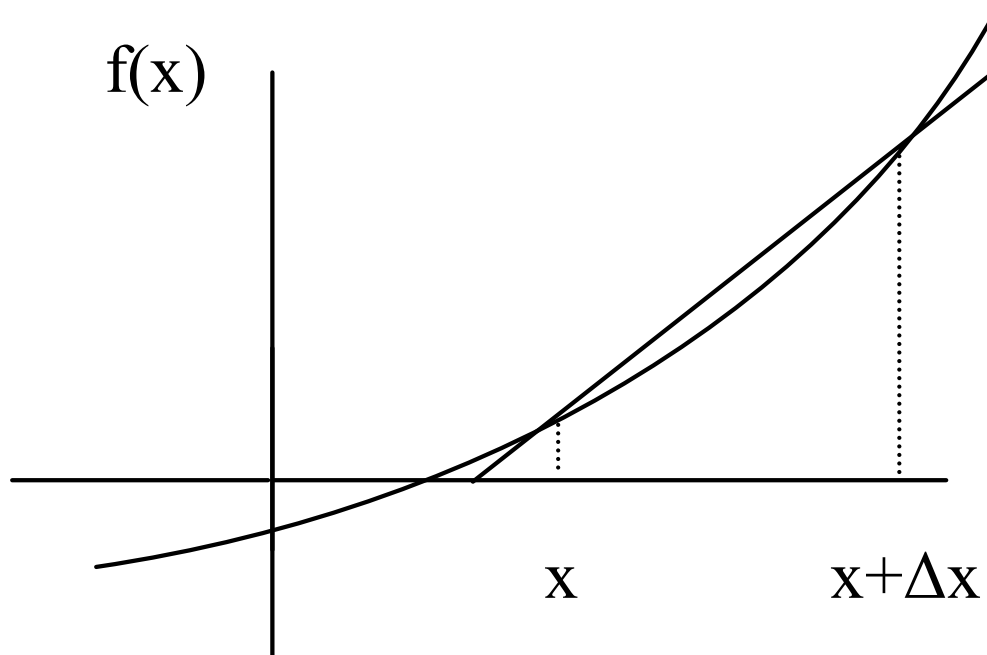


Figure 1: Graphical Representation of forward difference approximation of first derivative



Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

Using forward divided difference, find the acceleration of the rocket at $t=16s$.

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Table 1: Velocity as a function of time



Example 1 Cont.

$$a(t_i) \cong \frac{v(t_{i+1}) - v(t_i)}{\Delta t}$$

$$t_i = 15$$

$$t_{i+1} = t_i + \Delta t$$

$$t_{i+1} = 15 + 5 = 20$$

$$\begin{aligned} a(16) &\approx \frac{v(20) - v(15)}{5} \\ &= \frac{517.35 - 362.78}{5} \\ &= 30.914 \text{ m / s}^2 \end{aligned}$$



Direct Fit Polynomials

In this method, given ' $n+1$ ' data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$

, one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n$$

To find the first derivative,

$$P'_n(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2x + \dots + (n-1)a_{n-1}x^{n-2} + na_nx^{n-1}$$

Similarly other derivatives can be found.



Example 2-Direct Fit Polynomials

The upward velocity of a rocket is given as a function of time in Table 1

Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t=16s$.

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Table 2: Velocity as a function of time



Example 2-Direct Fit Polynomials cont.

For the third order polynomial (also called cubic interpolation), we choose the velocity given by

$$v(t) = a_0 + a_1t + a_2t^2 + a_3t^3$$

Since we want to find the velocity at $t=16$, and we are using third order polynomial, we need to choose the four points closest to $t=16$ and that also bracket $t=16$ to evaluate it.

The four points are $t_0=10$, $t_1=15$, $t_2=20$ and $t_3=22.5$.

$$t_0 = 10, \quad v(t_0) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

$$t_2 = 20, \quad v(t_2) = 517.35$$

$$t_3 = 22.5, \quad v(t_3) = 602.97$$

Example 2-Direct Fit Polynomials cont.

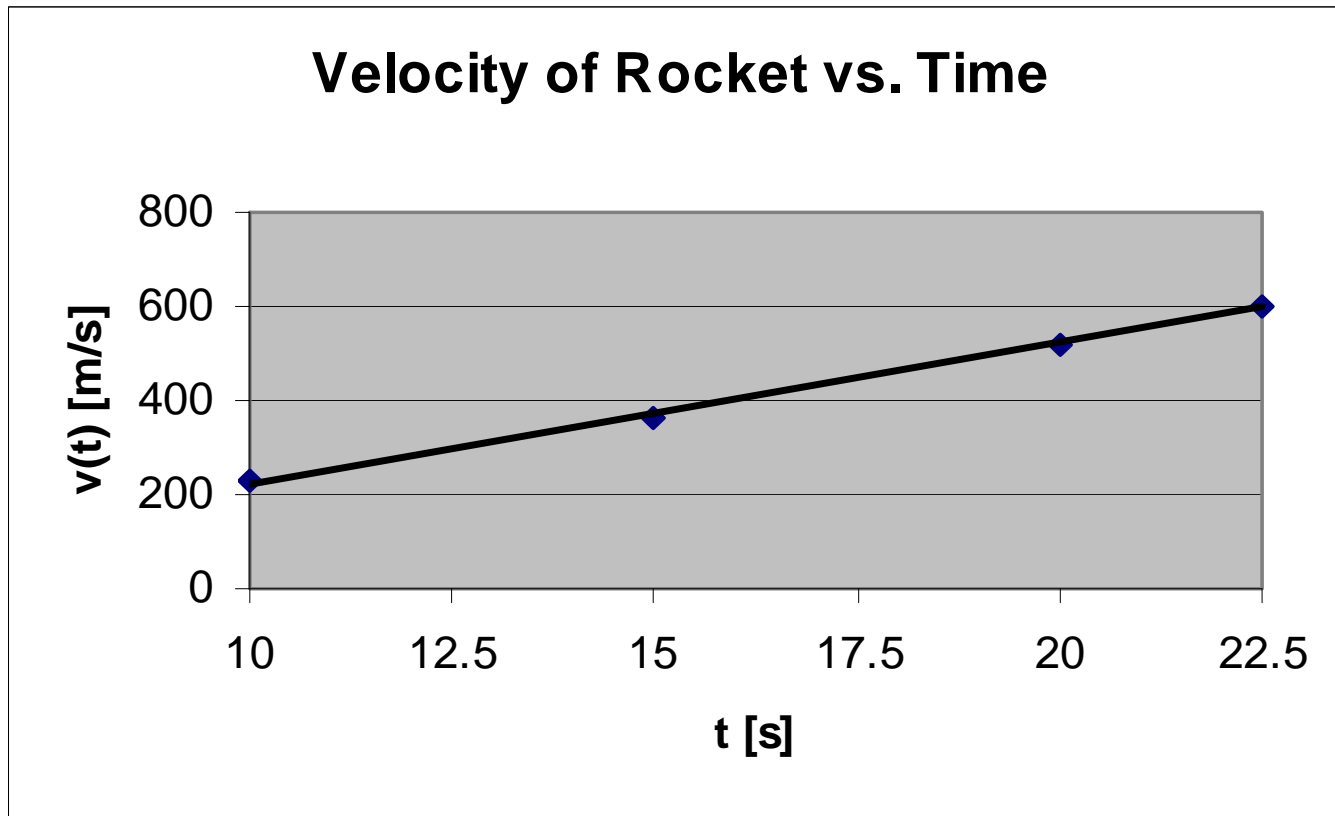


Figure 1: Graph of upward velocity of the rocket vs. time



Example 2-Direct Fit Polynomials cont.

such that

$$v(10) = 227.04 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$

$$v(15) = 362.78 = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3$$

$$v(20) = 517.35 = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3$$

$$v(22.5) = 602.97 = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3$$

Writing the four equations in matrix form, we have



Example 2-Direct Fit Polynomials cont.

$$\begin{bmatrix} 1 & 10 & 100 & 1000 \\ 1 & 15 & 225 & 3375 \\ 1 & 20 & 400 & 8000 \\ 1 & 22.5 & 506.25 & 11391 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 227.04 \\ 362.78 \\ 517.35 \\ 602.97 \end{bmatrix}$$

Solving the above four equations gives

$$a_0 = -4.3810$$

$$a_1 = 21.289$$

$$a_2 = 0.13065$$

$$a_3 = 0.0054606$$



Example 2-Direct Fit Polynomials cont.

Hence

$$\begin{aligned}v(t) &= a_0 + a_1t + a_2t^2 + a_3t^3 \\ &= -4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3, \quad 10 \leq t \leq 22.5\end{aligned}$$

The acceleration at $t=16$ is given by,

$$a(16) = \left. \frac{d}{dt} v(t) \right|_{t=16}$$

Given that

$$v(t) = -4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3, \quad 10 \leq t \leq 22.5$$

$$\begin{aligned}a(t) &= \frac{d}{dt} v(t) \\ &= \frac{d}{dt} (-4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3) \\ &= 21.289 + 0.26130t + 0.016382t^2, \quad 10 \leq t \leq 22.5\end{aligned}$$



Example 2-Direct Fit Polynomials cont.

$$\begin{aligned}a(16) &= 21.289 + 0.26130(16) + 0.016382(16)^2 \\ &= 29.664 \text{ m / s}^2\end{aligned}$$



Lagrange Polynomial

In this method, given $(x_1, y_1), \dots, (x_n, y_n)$, one can fit a $(n-1)^{th}$ order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.



Lagrange Polynomial Cont.

Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through

$(x_0, y_0), (x_1, y_1), (x_2, y_2)$ is

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating equation (2) gives



Lagrange Polynomial Cont.

$$f_2'(x) = \frac{2x - (x_1 + x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - (x_0 + x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2x - (x_0 + x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$



Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Determine the value of the acceleration at $t=16$ seconds using second order Lagrangian polynomial interpolation for velocity

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Table 3: Velocity as a function of time



Example 3 Cont.

Solution:

$$v(t) = \left(\frac{t-t_1}{t_0-t_1} \right) \left(\frac{t-t_2}{t_0-t_2} \right) v(t_0) + \left(\frac{t-t_0}{t_1-t_0} \right) \left(\frac{t-t_2}{t_1-t_2} \right) v(t_1) + \left(\frac{t-t_0}{t_2-t_0} \right) \left(\frac{t-t_1}{t_2-t_1} \right) v(t_2)$$

$$a(t) = \frac{2t - (t_1 + t_2)}{(t_0 - t_1)(t_0 - t_2)} v(t_0) + \frac{2t - (t_0 + t_2)}{(t_1 - t_0)(t_1 - t_2)} v(t_1) + \frac{2t - (t_0 + t_1)}{(t_2 - t_0)(t_2 - t_1)} v(t_2)$$

$$a(16) = \frac{2(16) - (15 + 20)}{(10 - 15)(10 - 20)} (227.04) + \frac{2(16) - (10 + 20)}{(15 - 10)(15 - 20)} (362.78) + \frac{2(16) - (10 + 15)}{(20 - 10)(20 - 15)} (517.35)$$

$$= -0.06(227.04) - 0.08(362.78) + 0.14(517.35)$$

$$= 29.784 \text{ m / s}^2$$