



# Simultaneous Linear Equations

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Topic: Gaussian Elimination  
Major: Industrial Engineering



# Gaussian Elimination

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One of the most popular techniques for solving simultaneous linear equations of the form  $[A][X] = [C]$

Consists of 2 steps

1. Forward Elimination of Unknowns.
2. Back Substitution



# Forward Elimination

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The goal of Forward Elimination is to transform the coefficient matrix into an Upper Triangular Matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$



# Forward Elimination

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## Linear Equations

A set of  $n$  equations and  $n$  unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n$$



# Forward Elimination

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## Transform to an Upper Triangular Matrix

Step 1: Eliminate  $x_1$  in 2<sup>nd</sup> equation using equation 1 as the pivot equation

$$\left[ \frac{\text{Eqn1}}{a_{11}} \right] \times (a_{21})$$

Which will yield

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \dots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$



# Forward Elimination

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Zeroing out the coefficient of  $x_1$  in the 2<sup>nd</sup> equation.

Subtract this equation from 2<sup>nd</sup> equation

$$\left( a_{22} - \frac{a_{21}}{a_{11}} a_{12} \right) x_2 + \dots + \left( a_{2n} - \frac{a_{21}}{a_{11}} a_{1n} \right) x_n = b_2 - \frac{a_{21}}{a_{11}} b_1$$

Or

$$a'_{22} x_2 + \dots + a'_{2n} x_n = b'_2$$

Where

$$a'_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

⋮

$$a'_{2n} = a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}$$



# Forward Elimination

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Repeat this procedure for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2$$

$$a'_{32}x_2 + a'_{33}x_3 + \dots + a'_{3n}x_n = b'_3$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$a'_{n2}x_2 + a'_{n3}x_3 + \dots + a'_{nn}x_n = b'_n$$



# Forward Elimination

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Step 2: Eliminate  $x_2$  in the 3<sup>rd</sup> equation.

Equivalent to eliminating  $x_1$  in the 2<sup>nd</sup> equation using equation 2 as the pivot equation.

$$Eqn3 - \left[ \frac{Eqn2}{a_{22}} \right] \times (a_{32})$$



# Forward Elimination

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This procedure is repeated for the remaining equations to reduce the set of equations as

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n &= b'_2 \\a''_{33}x_3 + \dots + a''_{3n}x_n &= b''_3 \\&\vdots \\a''_{n3}x_3 + \dots + a''_{nn}x_n &= b''_n\end{aligned}$$



# Forward Elimination

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Continue this procedure by using the third equation as the pivot equation and so on.

At the end of (n-1) Forward Elimination steps, the system of equations will look like:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \dots + a''_{nn}x_n = b''_3$$

$$\vdots$$

$$a^{(n-1)}_{nn}x_n = b^{(n-1)}_n$$

# Forward Elimination

At the end of the Forward Elimination steps

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ & & a''_{33} & \cdots & a''_{3n} \\ & & & \vdots & \vdots \\ & & & & a^{(n-1)}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b^{(n-1)}_n \end{bmatrix}$$



# Back Substitution

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The goal of Back Substitution is to solve each of the equations using the upper triangular matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Example of a system of 3 equations



# Back Substitution

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Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Solve the second from last equation  $(n-1)^{\text{th}}$  using  $x_n$  solved for previously.

This solves for  $x_{n-1}$ .



# Back Substitution

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Representing Back Substitution for all equations by formula

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^n a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \quad \text{For } i=n-1, n-2, \dots, 1$$

and

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

# Example: Production Optimization

To find the number of toys a company should manufacture per day to optimally use their injection-molding machine and the assembly line, one needs to solve the following set of equations. The unknowns are the number of toys for boys,  $x_1$ , number of toys for girls,  $x_2$ , and the number of unisexual toys,  $x_3$ .

$$\begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0.1667 & 0.6667 & 0.3333 \\ 1.05 & -1.00 & 0.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{Bmatrix} 756 \\ 1260 \\ 0 \end{Bmatrix}$$

Find the values of  $x_1$ ,  $x_2$ , and  $x_3$  using Naïve Gauss Elimination.

# Example: Production Optimization

Forward Elimination: Step 1

$$\text{Row2} - \left[ \frac{\text{Row1}}{0.3333} \right] \times (0.1667) =$$

Yields

$$\begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0 & 0.5833 & -0.0002 \\ 1.05 & -1.00 & 0.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 756 \\ 881.89 \\ 0 \end{bmatrix}$$

# Example: Production Optimization

Forward Elimination: Step 1

$$\text{Row3} - \left[ \frac{\text{Row1}}{0.3333} \right] \times (1.05) =$$

Yields

$$\begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0 & 0.5833 & -0.0002 \\ 0 & -1.52516 & -2.1003 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 756 \\ 881.89 \\ -2381.638 \end{bmatrix}$$

# Example: Production Optimization

Forward Elimination: Step 2

$$\text{Row3} - \left[ \frac{\text{Row2}}{0.5833} \right] \times (-1.52516) =$$

Yields

$$\begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0 & 0.5833 & -0.0002 \\ 0 & 0 & -2.1008 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 756 \\ 881.89 \\ -75.7519 \end{bmatrix}$$

This is now ready for Back Substitution



# Example: Production Optimization

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Back Substitution: Solve for  $x_3$  using the third equation

$$-2.1008x_3 = -75.7519$$

$$x_3 = \frac{-75.7519}{-2.1008}$$
$$= 36.0586$$



# Example: Production Optimization

Back Substitution: Solve for  $x_2$  using the second equation

$$0.5833x_2 + (-0.0002)x_3 = 881.89$$

$$\begin{aligned}x_2 &= \frac{881.89 - (-0.0002)x_3}{0.5833} \\ &= \frac{881.89 - (-0.0002) \times 36.0586}{0.5833} \\ &= 1511.91\end{aligned}$$

# Example: Production Optimization

Back Substitution: Solve for  $x_1$  using the first equation

$$0.3333x_1 + 0.1667x_2 + 0.6667x_3 = 756$$

$$x_1 = \frac{756 - 0.1667x_2 - 0.6667x_3}{0.3333}$$

$$= \frac{756 - 0.1667 \times 1511.91 - 0.6667 \times 36.0586}{0.3333}$$

$$= 1439.92$$

# Example: Production Optimization

Solution:

The solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1439.92 \\ 1511.91 \\ 36.0586 \end{bmatrix}$$

1440 toys for boys should be produced

1512 toys for girls should be produced

36 unisex toys should be produced



# Pitfalls

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## Two Potential Pitfalls

-Division by zero: May occur in the forward elimination steps. Consider the set of equations:

$$10x_2 - 7x_3 = 7$$

$$6x_1 + 2.099x_2 + 3x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

- Round-off error: Prone to round-off errors.



# Pitfalls: Example

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Consider the system of equations:

Use five significant figures with chopping

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

At the end of Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$



# Pitfalls: Example

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## Back Substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

$$x_3 = \frac{15004}{15005} = 0.99993$$

$$x_2 = \frac{6.001 - 6x_3}{-0.001} = -1.5$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = -0.3500$$



# Pitfalls: Example

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Compare the calculated values with the exact solution

$$[X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$



# Improvements

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## Increase the number of significant digits

Decreases round off error

Does not avoid division by zero

## Gaussian Elimination with Partial Pivoting

Avoids division by zero

Reduces round off error



# Partial Pivoting

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Gaussian Elimination with partial pivoting applies row switching to normal Gaussian Elimination.

How?

At the beginning of the  $k^{\text{th}}$  step of forward elimination, find the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \dots, |a_{nk}|$$

If the maximum of the values is  $|a_{pk}|$  in the  $p^{\text{th}}$  row,  $k \leq p \leq n$ ,

then switch rows  $p$  and  $k$ .



# Partial Pivoting

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What does it Mean?

Gaussian Elimination with Partial Pivoting ensures that each step of Forward Elimination is performed with the pivoting element  $|a_{kk}|$  having the largest absolute value.



# Partial Pivoting: Example

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Consider the system of equations

$$10x_1 - 7x_2 = 7$$

$$-3x_1 + 2.099x_2 + 3x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

# Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

$|10|$ ,  $|-3|$ , and  $|5|$  or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

# Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

$|-0.001|$  and  $|2.5|$  or  $0.0001$  and  $2.5$

The largest absolute value is  $2.5$ , so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$



# Partial Pivoting: Example

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Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

# Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_3}{2.5} = 1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$



# Partial Pivoting: Example

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Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{\text{calculated}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad [X]_{\text{exact}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$



# Summary

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- Forward Elimination
- Back Substitution
- Pitfalls
- Improvements
- Partial Pivoting