Chapter 04.08
Gauss-Seidel Method

After reading this chapter, you should be able to:

1. *solve a set of equations using the Gauss-Seidel method,*
2. *recognize the advantages and pitfalls of the Gauss-Seidel method, and*
3. *determine under what conditions the Gauss-Seidel method always converges.*

Why do we need another method to solve a set of simultaneous linear equations?

In certain cases, such as when a system of equations is large, iterative methods of solving equations are more advantageous. Elimination methods, such as Gaussian elimination, are prone to large round-off errors for a large set of equations. Iterative methods, such as the Gauss-Seidel method, give the user control of the round-off error. Also, if the physics of the problem are well known, initial guesses needed in iterative methods can be made more judiciously leading to faster convergence.

What is the algorithm for the Gauss-Seidel method? Given a general set of n equations and n unknowns, we have

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2 \\
&. \qquad\qquad . \\
&. \qquad\qquad . \\
&. \qquad\qquad . \\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = c_n
\end{aligned}$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with x_1 on the left hand side, the second equation is rewritten with x_2 on the left hand side and so on as follows

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

These equations can be rewritten in a summation form as

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j \neq 1}}^{n} a_{1j}x_j}{a_{11}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j \neq 2}}^{n} a_{2j}x_j}{a_{22}}$$

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$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j \neq n-1}}^{n} a_{n-1,j}x_j}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j \neq n}}^{n} a_{nj}x_j}{a_{nn}}$$

Hence for any row i,

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}x_j}{a_{ii}}, i = 1,2,\ldots,n.$$

Now to find x_i's, one assumes an initial guess for the x_i's and then uses the rewritten equations to calculate the new estimates. Remember, one always uses the most recent estimates to calculate the next estimates, x_i. At the end of each iteration, one calculates the absolute relative approximate error for each x_i as

$$\left|\in_a\right|_i = \left|\frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}}\right| \times 100$$

where x_i^{new} is the recently obtained value of x_i, and x_i^{old} is the previous value of x_i.

When the absolute relative approximate error for each x_i is less than the pre-specified tolerance, the iterations are stopped.

Example 1

To find the number of toys a company should manufacture per day to optimally use their injection-molding machine and the assembly line, one needs to solve the following set of equations. The unknowns are the number of toys for boys, x_1, the number of toys for girls, x_2, and the number of unisexual toys, x_3.

$$\begin{bmatrix} 0.3333 & 0.1667 & 0.6667 \\ 0.1667 & 0.6667 & 0.3333 \\ 1.05 & -1.00 & 0.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 756 \\ 1260 \\ 0 \end{bmatrix}$$

Find the values of x_1, x_2, and x_3 using the Gauss-Seidel method. Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution

Rewriting the equations gives

$$x_1 = \frac{756 - 0.1667x_2 - 0.6667x_3}{0.3333}$$

$$x_2 = \frac{1260 - 0.1667x_1 - 0.3333x_3}{0.6667}$$

$$x_3 = \frac{0 - 1.05x_1 - (-1.00)x_2}{0}$$

The equation for x_3 is divided by 0 which is undefined. Therefore the order of the equations will need to be changed. Equation 3 and Equation 1 will be switched. By switching Equations 3 and 1, the matrix will also become diagonally dominant.

The system of equations becomes

$$\begin{bmatrix} 1.05 & -1.00 & 0.00 \\ 0.1667 & 0.6667 & 0.3333 \\ 0.3333 & 0.1667 & 0.6667 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1260 \\ 756 \end{bmatrix}$$

Rewriting the equations gives

$$x_1 = \frac{0 - (-1.00)x_2 - 0x_3}{1.05}$$

$$x_2 = \frac{1260 - 0.1667x_1 - 0.3333x_3}{0.6667}$$

$$x_3 = \frac{756 - 0.3333x_1 - 0.1667x_2}{0.6667}$$

Iteration #1

Given the initial guess of the solution vector as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \\ 100 \end{bmatrix}$$

we get

$$x_1 = \frac{0-(-1.00)\times 1000 - 0\times 100}{1.05}$$

$$= 952.38$$

$$x_2 = \frac{1260 - 0.1667\times 952.38 - 0.3333\times 100}{0.6667}$$

$$= 1601.8$$

$$x_3 = \frac{756 - 0.3333\times 952.38 - 0.1667\times 1601.8}{0.6667}$$

$$= 257.32$$

The absolute relative approximate error for each x_i then is

$$|\in_a|_1 = \left|\frac{952.38-1000}{952.38}\right|\times 100$$

$$= 5\%$$

$$|\in_a|_2 = \left|\frac{1601.8-1000}{1601.8}\right|\times 100$$

$$= 37.570\%$$

$$|\in_a|_3 = \left|\frac{257.32-100}{257.32}\right|\times 100$$

$$= 61.138$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 952.38 \\ 1601.8 \\ 257.32 \end{bmatrix}$$

and the maximum absolute relative approximate error is 61.138%.

Iteration #2

The estimate of the solution vector at the end of Iteration #1 is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 952.38 \\ 1601.8 \\ 257.32 \end{bmatrix}$$

Now we get

$$x_1 = \frac{0-(-1.00)\times 1601.8 - 0\times 257.32}{1.05}$$
$$= 1525.5$$
$$x_2 = \frac{1260 - 0.1667\times 1525.5 - 0.3333\times 257.32}{0.6667}$$
$$= 1379.8$$
$$x_3 = \frac{756 - 0.3333\times 1525.5 - 0.1667\times 1379.8}{0.6667}$$
$$= 26.295$$

The absolute relative approximate error for each x_i then is

$$\left|\in_a\right|_1 = \left|\frac{1525.5 - 952.38}{1525.5}\right|\times 100$$
$$= 37.570\%$$
$$\left|\in_a\right|_2 = \left|\frac{1379.8 - 1601.8}{1379.8}\right|\times 100$$
$$= 16.085\%$$
$$\left|\in_a\right|_3 = \left|\frac{26.295 - 257.32}{26.295}\right|\times 100$$
$$= 878.59\%$$

At the end of the second iteration, the estimate of the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1525.5 \\ 1379.8 \\ 26.295 \end{bmatrix}$$

and the maximum absolute relative approximate error is 878.59%.

Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
1	952.38	5	1601.8	37.570	257.32	61.138
2	1525.5	37.570	1379.8	16.085	26.295	878.59
3	1314.1	16.085	1548.2	10.874	89.876	70.743
4	1474.5	10.874	1476.3	4.8686	27.694	224.53
5	1406.0	4.8686	1524.5	3.1618	49.863	44.459
6	1451.9	3.1618	1501.9	1.5021	32.554	53.170

After six iterations, the absolute relative approximate errors are decreasing, but they are still high. Allowing for more iterations, the absolute relative approximate errors decrease significantly.

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
20	1439.8	0.00064276	1511.8	0.00034987	36.115	0.0091495
21	1439.8	0.00034987	1511.8	0.00019257	36.114	0.0049578

This is close to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1439.8 \\ 1511.8 \\ 36.113 \end{bmatrix}$$

Example 2

Find the solution to the following system of equations using the Gauss-Seidel method.

$$\begin{aligned} 12x_1 + 3x_2 - 5x_3 &= 1 \\ x_1 + 5x_2 + 3x_3 &= 28 \\ 3x_1 + 7x_2 + 13x_3 &= 76 \end{aligned}$$

Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution

The coefficient matrix

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidel method.

Rewriting the equations, we get

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Iteration #1

$$x_1 = \frac{1-3(0)+5(1)}{12}$$

$$= 0.50000$$

$$x_2 = \frac{28-(0.50000)-3(1)}{5}$$

$$= 4.9000$$

$$x_3 = \frac{76-3(0.50000)-7(4.9000)}{13}$$

$$= 3.0923$$

The absolute relative approximate error at the end of the first iteration is

$$\left|\in_a\right|_1 = \left|\frac{0.50000-1}{0.50000}\right| \times 100$$

$$= 100.000\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.9000-0}{4.9000}\right| \times 100$$

$$= 100.000\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.0923-1}{3.0923}\right| \times 100$$

$$= 67.662\%$$

The maximum absolute relative approximate error is 100.000%

Iteration #2

$$x_1 = \frac{1-3(4.9000)+5(3.0923)}{12}$$

$$= 0.14679$$

$$x_2 = \frac{28-(0.14679)-3(3.0923)}{5}$$

$$= 3.7153$$

$$x_3 = \frac{76-3(0.14679)-7(3.7153)}{13}$$

$$= 3.8118$$

At the end of second iteration, the absolute relative approximate error is

$$\left|\in_a\right|_1 = \left|\frac{0.14679-0.50000}{0.14679}\right| \times 100$$

$$= 240.61\%$$

$$\left|\in_a\right|_2 = \left|\frac{3.7153-4.9000}{3.7153}\right| \times 100$$

$$= 31.889\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.8118-3.0923}{3.8118}\right| \times 100$$

$$= 18.874\%$$

The maximum absolute relative approximate error is 240.61%. This is greater than the value of 100.00% we obtained in the first iteration. Is the solution diverging? No, as you conduct more iterations, the solution converges as follows.

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
1	0.50000	100.00	4.9000	100.00	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.874
3	0.74275	80.236	3.1644	17.408	3.9708	4.0064
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

This is close to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Example 3

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

find the solution using the Gauss-Seidel method. Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as the initial guess.

Solution

Rewriting the equations, we get

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

the next six iterative values are given in the table below.

Iteration	x_1	$\left\vert \in_a \right\vert_1 \%$	x_2	$\left\vert \in_a \right\vert_2 \%$	x_3	$\left\vert \in_a \right\vert_3 \%$
1	21.000	95.238	0.80000	100.00	50.680	98.027
2	–196.15	110.71	14.421	94.453	–462.30	110.96
3	1995.0	109.83	–116.02	112.43	4718.1	109.80
4	–20149	109.90	1204.6	109.63	–47636	109.90
5	2.0364×10^5	109.89	–12140	109.92	4.8144×10^5	109.89
6	-2.0579×10^6	109.89	1.2272×10^5	109.89	-4.8653×10^6	109.89

You can see that this solution is not converging and the coefficient matrix is not diagonally dominant. The coefficient matrix

$$[A]=\begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

is not diagonally dominant as

$$|a_{11}| = |3| = 3 \le |a_{12}| + |a_{13}| = |7| + |13| = 20$$

Hence, the Gauss-Seidel method may or may not converge.

However, it is the same set of equations as the previous example and that converged. The only difference is that we exchanged first and the third equation with each other and that made the coefficient matrix not diagonally dominant.

Therefore, it is possible that a system of equations can be made diagonally dominant if one exchanges the equations with each other. However, it is not possible for all cases. For example, the following set of equations

$$x_1 + x_2 + x_3 = 3$$

$$2x_1 + 3x_2 + 4x_3 = 9$$

$$x_1 + 7x_2 + x_3 = 9$$

cannot be rewritten to make the coefficient matrix diagonally dominant.

SIMULTANEOUS LINEAR EQUATIONS	
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