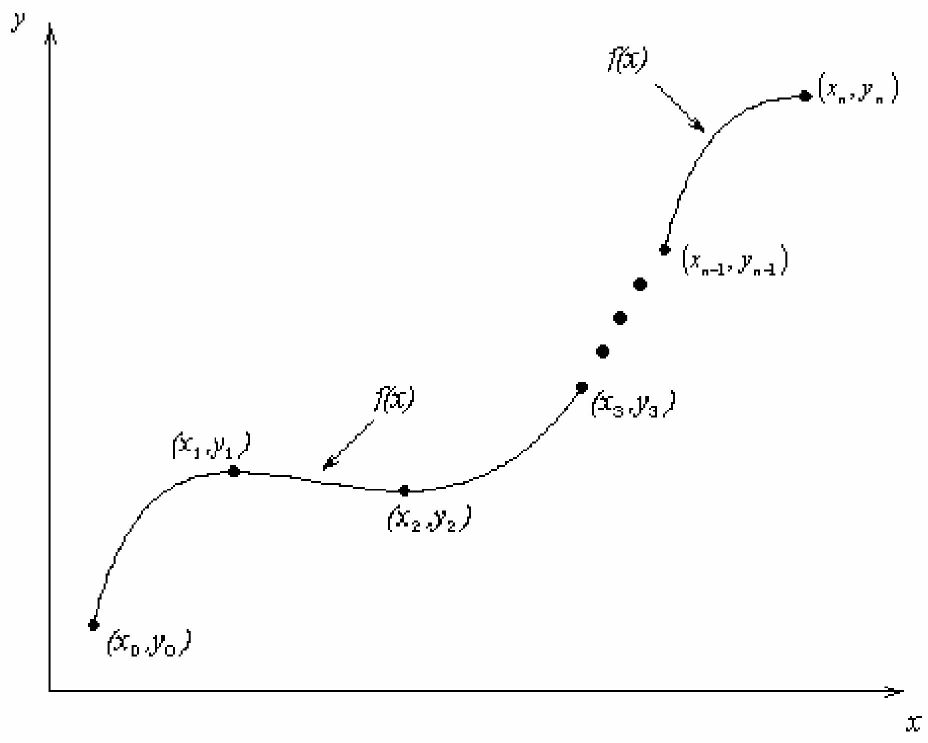
Interpolation

Topic: Direct Method

Major: Industrial

What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Direct Method

Given 'n+1' data points (x_0,y_0), (x_1,y_1),………… (x_n,y_n), pass a polynomial of order 'n' through the data as given below:

$$y = a_0 + a_1 x + \dots\dots\dots\dots\dots\dots + a_n x^n.$$

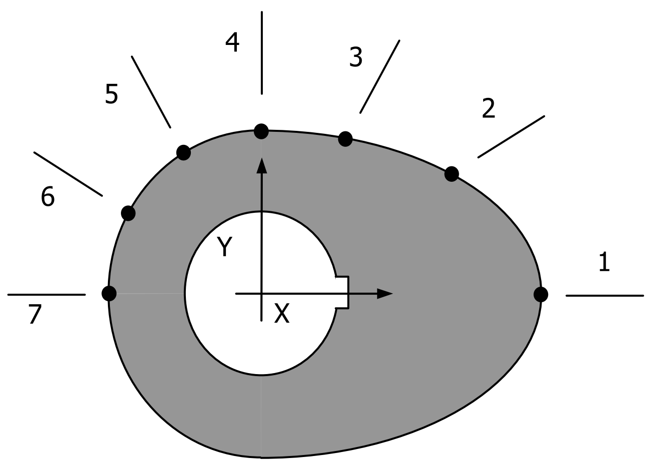
where a_0, a_1,……………… a_n are real constants.

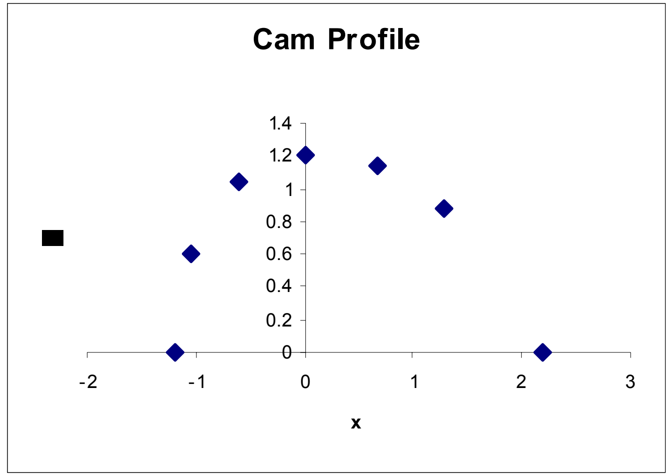
- Set up 'n+1' equations to find 'n+1' constants.
- To find the value 'y' at a given value of 'x', simply substitute the value of 'x' in the above polynomial.

Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the direct method.

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0



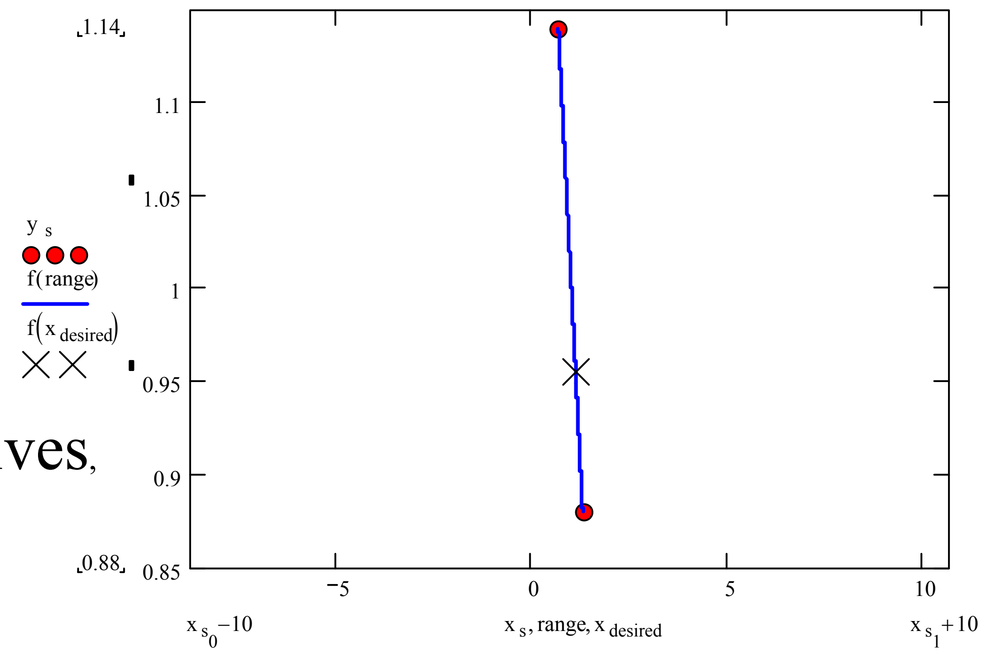


Linear Interpolation

$$y(x) = a_0 + a_1 x$$

$$y(1.28) = a_0 + a_1(1.28) = 0.88$$

$$y(0.66) = a_0 + a_1(0.66) = 1.14$$



Solving the above two equations gives,

$$a_0 = 1.4168 \qquad a_1 = -0.41935$$

Hence

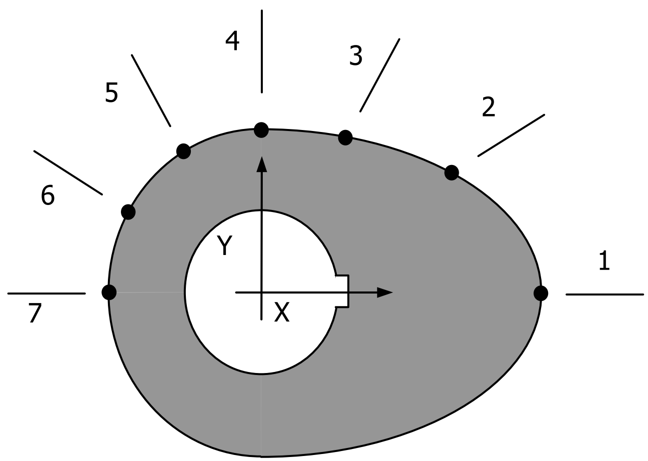
$$y(x) = 1.4168 - 0.41935x, \;\; 0.66 \le x \le 1.28.$$

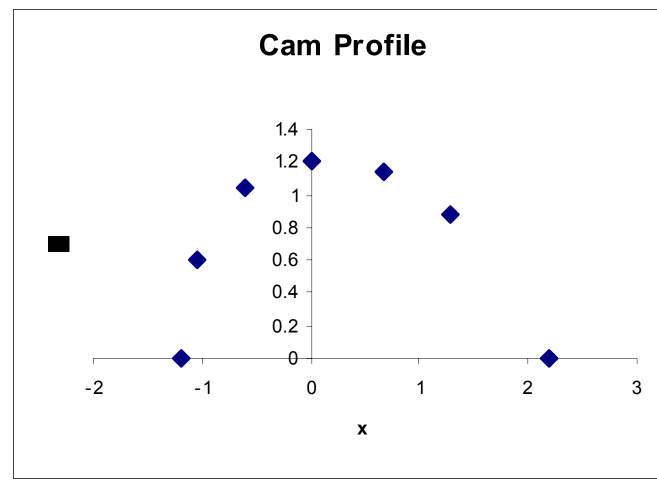
$$y(1.10) = 1.4168 - 0.41935(1.10) = 0.95548 \; in.$$

Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a quadratic path from x=2.20 to x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the direct method.

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0





Quadratic Interpolation

$$y(x) = a_0 + a_1 x + a_2 x^2$$

$$y(2.20) = a_0 + a_1(2.20) + a_2(2.20)^2 = 0$$

$$y(1.28) = a_0 + a_1(1.28) + a_2(1.28)^2 = 0.88$$

$$y(0.66) = a_0 + a_1(0.66) + a_2(0.66)^2 = 1.14$$

Solving the above three equations gives

$$a_0 = 1.1221 \qquad a_1 = 0.25734 \qquad a_2 = -0.34881$$

Quadratic Interpolation (contd)

$$y(x) = 1.1221 + 0.25734x - 0.34881x^2, \quad 0.66 \le x \le 2.20$$

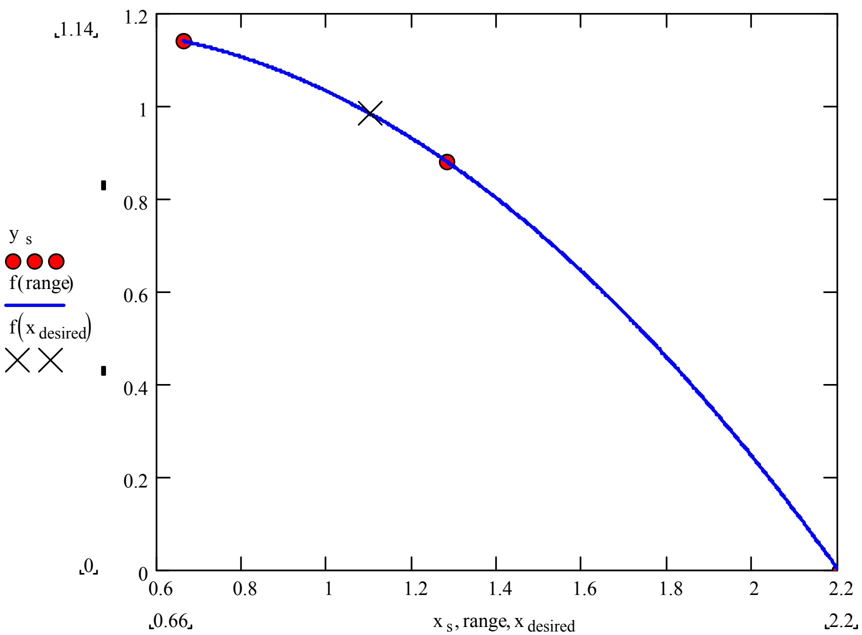
$$y(1.10) = 1.1221 + 0.25734(1.10) - 0.34881(1.10)^2$$

$$= 0.98311 \ in.$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left| \frac{0.98311 - 0.95548}{0.98311} \right| \times 100$$

$$= 2.810\%$$



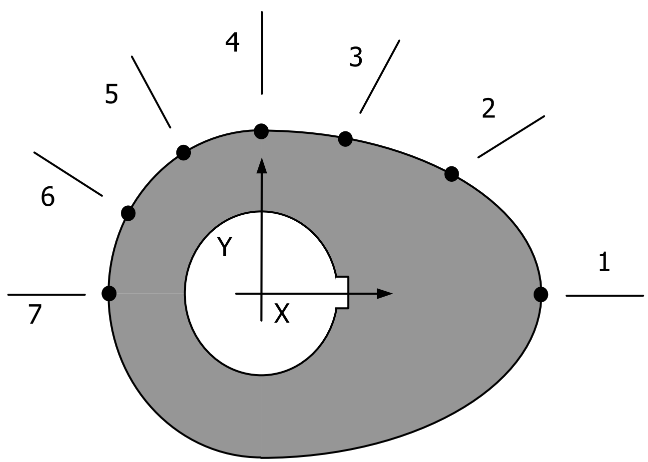
Comparison Table

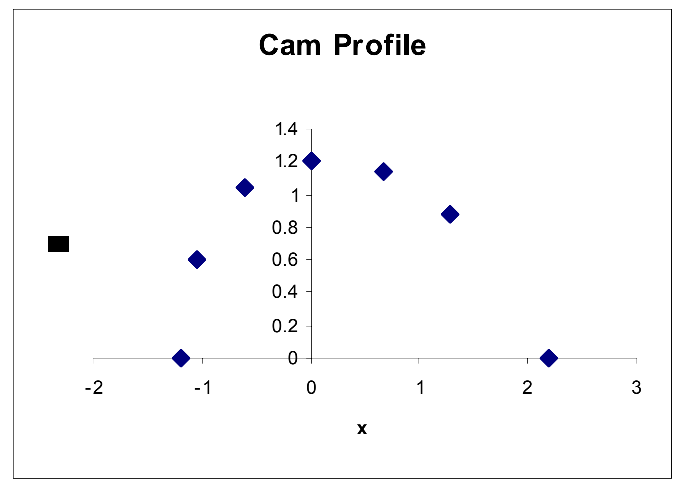
Order of Polynomial	1	2
Value of Y at X=1.1	0.98311	0.95548
Absolute Relative Approximate Error	----------	2.810 %

Example

A curve needs to be fit through the given points to fabricate the cam. Find the path using the direct method of interpolation.

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0





Sixth Order Interpolation

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$$

$$y(2.20) = 0.00 = a_0 + a_1(2.20) + a_2(2.20)^2 + a_3(2.20)^3 + a_4(2.20)^4 + a_5(2.20)^5 + a_6(2.20)^6$$

$$y(1.28) = 0.88 = a_0 + a_1(1.28) + a_2(1.28)^2 + a_3(1.28)^3 + a_4(1.28)^4 + a_5(1.28)^5 + a_6(1.28)^6$$

$$y(0.66) = 1.14 = a_0 + a_1(0.66) + a_2(0.66)^2 + a_3(0.66)^3 + a_4(0.66)^4 + a_5(0.66)^5 + a_6(0.66)^6$$

$$y(0.00) = 1.20 = a_0 + a_1(0.00) + a_2(0.00)^2 + a_3(0.00)^3 + a_4(0.00)^4 + a_5(0.00)^5 + a_6(0.00)^6$$

$$y(-0.60) = 1.04 = a_0 + a_1(-0.60) + a_2(-0.60)^2 + a_3(-0.60)^3 + a_4(-0.60)^4 + a_5(-0.60)^5 + a_6(-0.60)^6$$

$$y(-1.04) = 0.60 = a_0 + a_1(-1.04) + a_2(-1.04)^2 + a_3(-1.04)^3 + a_4(-1.04)^4 + a_5(-1.04)^5 + a_6(-1.04)^6$$

$$y(-1.20) = 0.00 = a_0 + a_1(-1.20) + a_2(-1.20)^2 + a_3(-1.20)^3 + a_4(-1.20)^4 + a_5(-1.20)^5 + a_6(-1.20)^6$$

Sixth Order Interpolation (contd)

Writing the seven equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.20 & 2.20^2 & 2.20^3 & 2.20^4 & 2.20^5 & 2.20^6 \\ 1 & 1.28 & 1.28^2 & 1.28^3 & 1.28^4 & 1.28^5 & 1.28^6 \\ 1 & 0.66 & 0.66^2 & 0.66^3 & 0.66^4 & 0.66^5 & 0.66^6 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -0.60 & 0.60^2 & -0.60^3 & 0.60^4 & -0.60^5 & 0.60^6 \\ 1 & -1.04 & 1.04^2 & -1.04^3 & 1.04^4 & -1.04^5 & 1.04^6 \\ 1 & -1.20 & 1.20^2 & -1.20^3 & 1.20^4 & -1.20^5 & 1.20^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.88 \\ 1.14 \\ 1.20 \\ 1.04 \\ 0.60 \\ 0.00 \end{bmatrix}$$

Sixth Order Polynomial (contd)

Solving the above seven equations gives

$a_0 = 1.2$ $a_2 = -0.272550$ $a_4 = 0.072013$ $a_6 = -0.171029$

$a_1 = 0.251115$ $a_3 = -0.567651$ $a_5 = 0.452405$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$$

$$= 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3$$

$$+ 0.072013x^4 + 0.45241x^5 - 0.17103x^6, \quad -1.20 \le x \le 2.20$$

Sixth Order Polynomial (contd)

$$y(x) = 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3$$
$$+ 0.072013x^4 + 0.45241x^5 - 0.17103x^6, \quad -1.20 \le x \le 2.20$$