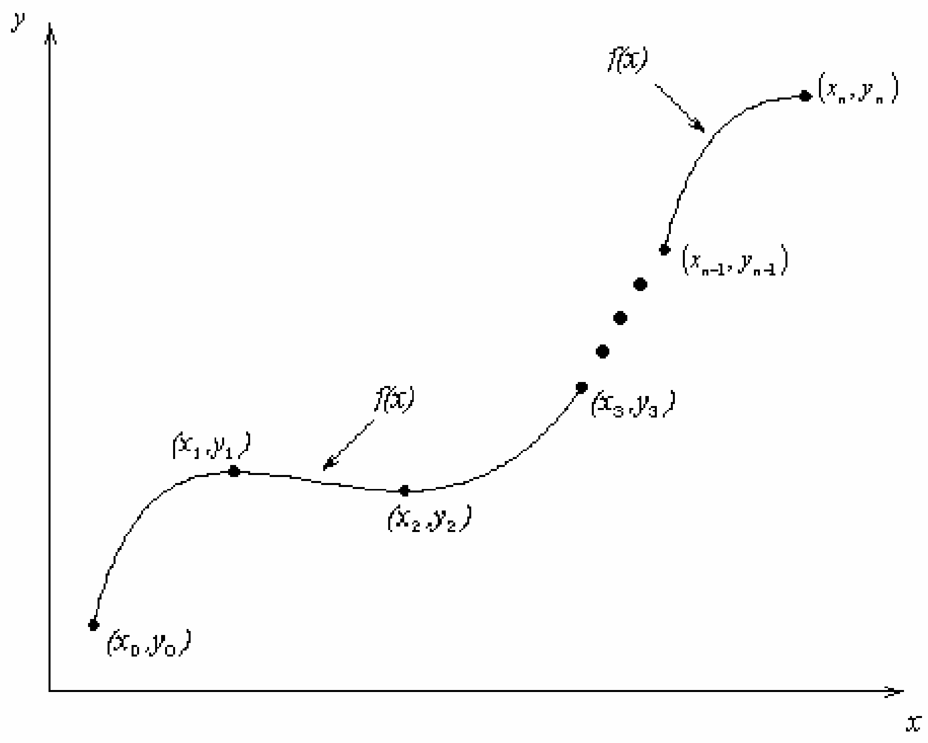
Interpolation

Topic: Lagrangian Interpolation

Major: Industrial

What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

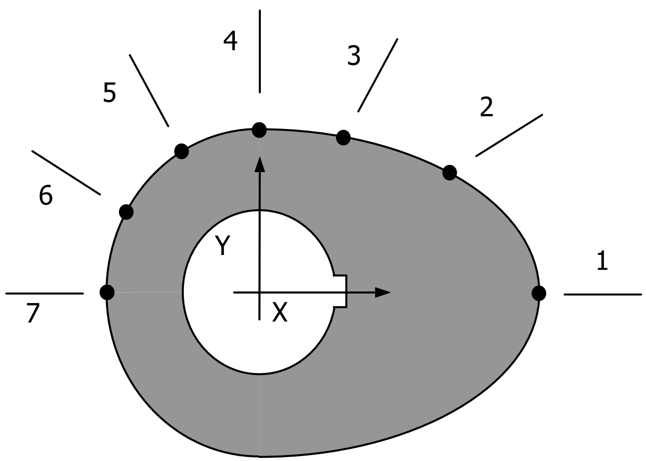
$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

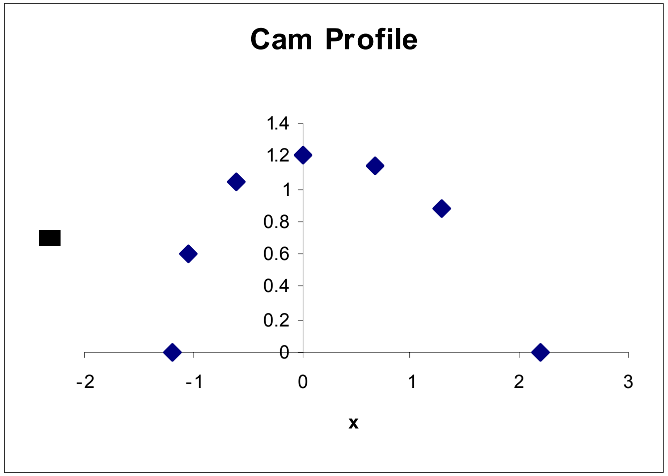
$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the Lagrangian method.

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0





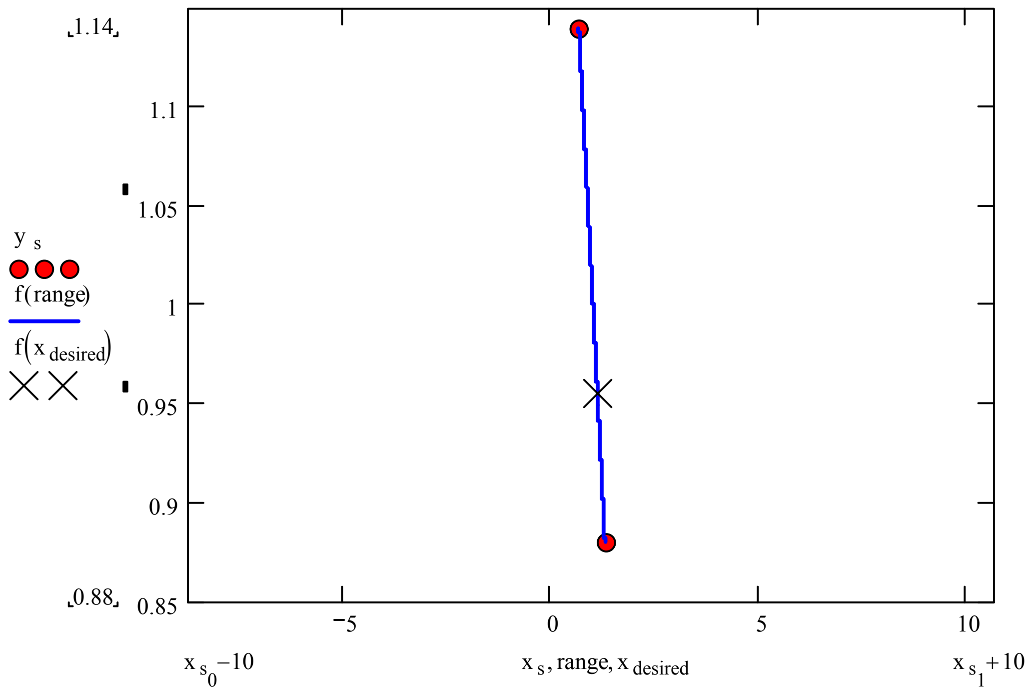
Linear Interpolation

$$y(x) = \sum_{i=0}^{1} L_i(x) y(x_i)$$

$$= L_0(x) y(x_0) + L_1(x) y(x_1)$$

$$x_0 = 1.28, y(x_0) = 0.88$$

$$x_1 = 0.66, y(x_1) = 1.14$$



Linear Interpolation (contd)

$$L_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{1} \frac{x - x_j}{x_0 - x_j} = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{1} \frac{x - x_j}{x_1 - x_j} = \frac{x - x_0}{x_1 - x_0}$$

$$y(x) = \frac{x - x_1}{x_0 - x_1} y(x_0) + \frac{x - x_0}{x_1 - x_0} y(x_1) = \frac{x - 0.66}{1.28 - 0.66}(0.88) + \frac{x - 1.28}{0.66 - 1.28}(1.14)$$

$$y(1.10) = \frac{1.10 - 0.66}{1.28 - 0.66}(0.88) + \frac{1.10 - 1.28}{0.66 - 1.28}(1.14)$$

$$= 0.70968(0.88) + 0.29032(1.14)$$

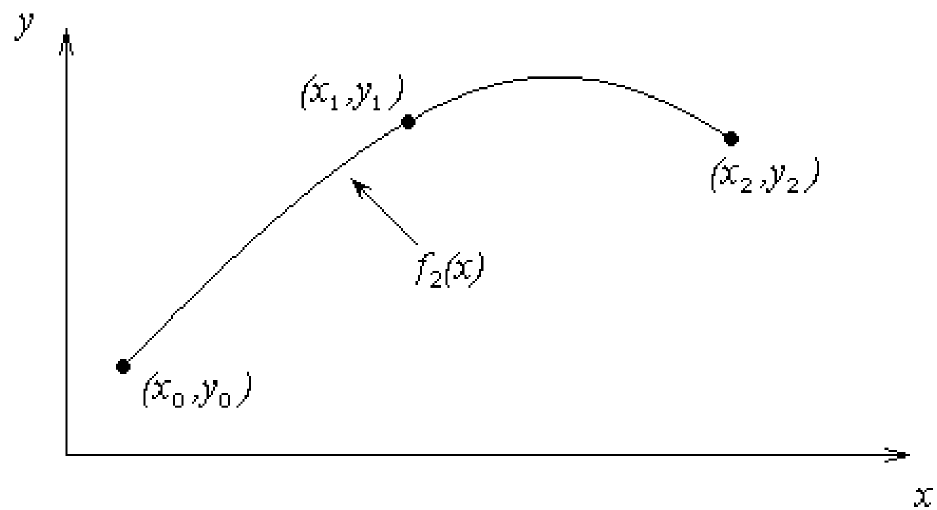
$$= 0.95548\ in.$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the value of y given by

$$y(x) = \sum_{i=0}^{2} L_i(x)\, y(x_i)$$

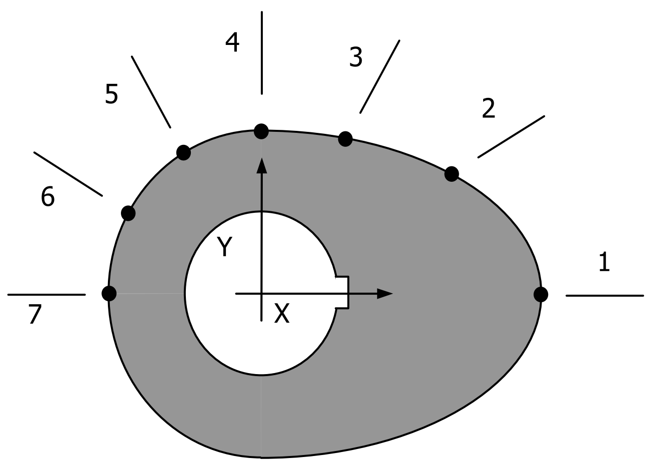
$$= L_0(x)\, y(x_0) + L_1(x)\, y(x_1) + L_2(x)\, y(x_2)$$

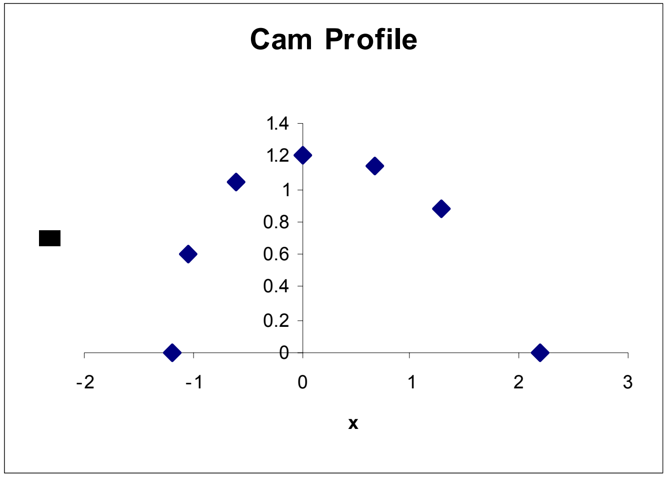


Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a quadratic profile from x =2.2 to x=1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the Lagrangian method.

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0



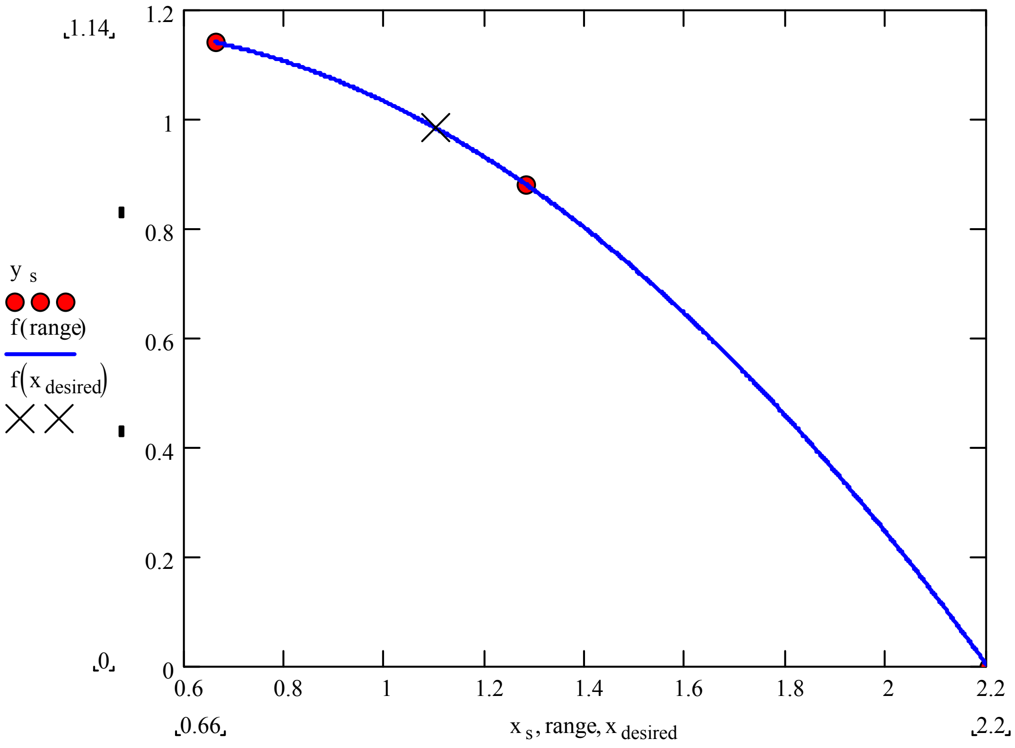


Quadratic Interpolation (contd)

$$x_o = 2.20, \quad y(x_o) = 0.00$$

$$x_1 = 1.28, \quad y(x_1) = 0.88$$

$$x_2 = 0.66, \quad y(x_2) = 1.14$$



$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{2} \frac{x - x_j}{x_0 - x_j} = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{2} \frac{x - x_j}{x_1 - x_j} = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j \neq 2}}^{2} \frac{x - x_j}{x_2 - x_j} = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)$$

Quadratic Interpolation (contd)

$$y(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)y(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)y(x_1) + \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)y(x_2)$$

$$y(1.10) = \frac{(1.10 - 1.28)(1.10 - 0.66)}{(2.20 - 1.28)(2.20 - 0.66)}(0.00) + \frac{(1.10 - 2.20)(1.10 - 0.66)}{(1.28 - 2.20)(1.28 - 0.66)}(0.88) + \frac{(1.10 - 2.20)(1.10 - 1.28)}{(0.66 - 2.20)(0.66 - 1.28)}(1.14)$$

$$= (-0.05590)(0.00) + (0.84853)(0.88) + (0.20737)(1.14)$$

$$= 0.98311 \text{ } in.$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{0.98311 - 0.95548}{0.98311}\right| \times 100$$

$$= 2.810\%$$

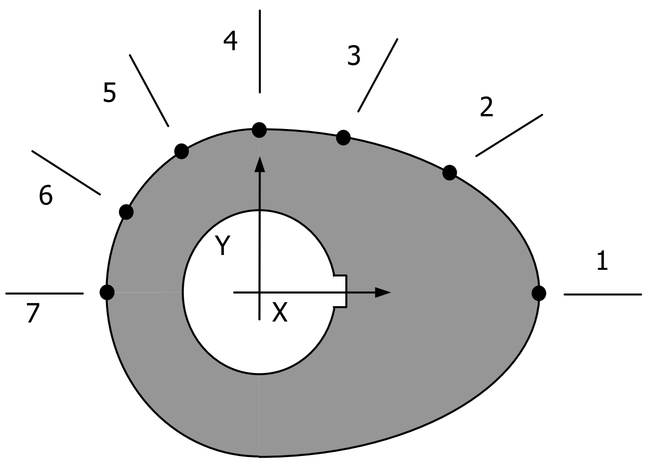
Comparison Table

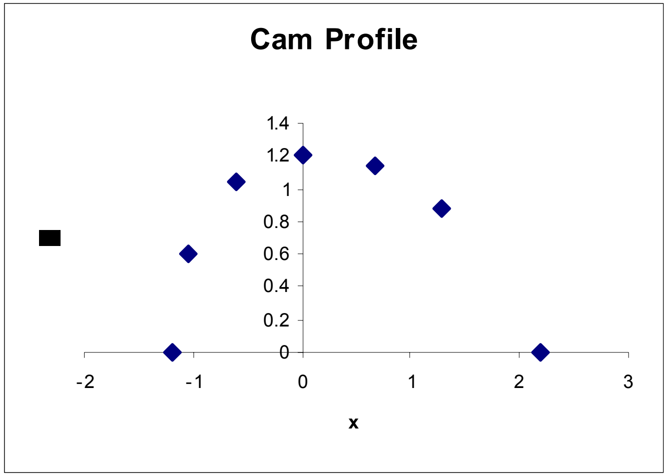
Order of Polynomial	1	2
Value of y at x=1.1	0.95548	0.98311
Absolute Relative Approximate Error	----------	2.810 %

Example

A curve needs to be fit through the seven points to fabricate the cam. Find the cam profile using all the seven points using Lagrangian method and sixth order polynomial interpolation

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0





Sixth Order Interpolation

$$y(x) = \sum_{i=0}^{6} L_i(x) y(x_i)$$

$$= L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) + L_3(x)y(x_3) + L_4(x)y(x_4) + L_5(x)y(x_5) + L_6(x)y(x_6)$$

$$x_o = 2.20, \quad y(x_o) = 0.00$$

$$x_1 = 1.28, \quad y(x_1) = 0.88$$

$$x_2 = 0.66, \quad y(x_2) = 1.14$$

$$x_3 = 0.00, \quad y(x_3) = 1.20$$

$$x_4 = -.60, \quad y(x_4) = 1.04$$

$$x_5 = -1.04, \quad y(x_5) = 0.60$$

$$x_6 = -1.20, \quad y(x_6) = 0.00$$

Sixth Order Interpolation (contd)

$$L_0(x)=\prod_{\substack{j=0\\ j\neq 0}}^{6}\frac{x-x_j}{x_0-x_j}=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)\left(\frac{x-x_4}{x_0-x_4}\right)\left(\frac{x-x_5}{x_0-x_5}\right)\left(\frac{x-x_6}{x_0-x_6}\right)$$

$$L_1(x)=\prod_{\substack{j=0\\ j\neq 1}}^{6}\frac{x-x_j}{x_1-x_j}=\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)\left(\frac{x-x_4}{x_1-x_4}\right)\left(\frac{x-x_5}{x_1-x_5}\right)\left(\frac{x-x_6}{x_1-x_6}\right)$$

$$L_2(x)=\prod_{\substack{j=0\\ j\neq 2}}^{6}\frac{x-x_j}{x_2-x_j}=\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)\left(\frac{x-x_6}{x_2-x_6}\right)$$

$$L_3(x)=\prod_{\substack{j=0\\ j\neq 3}}^{6}\frac{x-x_j}{x_3-x_j}=\left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)\left(\frac{x-x_6}{x_3-x_6}\right)$$

$$L_4(x)=\prod_{\substack{j=0\\ j\neq 4}}^{6}\frac{x-x_j}{x_4-x_j}=\left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)\left(\frac{x-x_6}{x_4-x_6}\right)$$

$$L_5(x)=\prod_{\substack{j=0\\ j\neq 5}}^{6}\frac{x-x_j}{x_5-x_j}=\left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)\left(\frac{x-x_6}{x_5-x_6}\right)$$

$$L_6(x)=\prod_{\substack{j=0\\ j\neq 6}}^{6}\frac{x-x_j}{x_6-x_j}=\left(\frac{x-x_0}{x_6-x_0}\right)\left(\frac{x-x_1}{x_6-x_1}\right)\left(\frac{x-x_2}{x_6-x_2}\right)\left(\frac{x-x_3}{x_6-x_3}\right)\left(\frac{x-x_4}{x_6-x_4}\right)\left(\frac{x-x_5}{x_6-x_5}\right)$$

Sixth Order Polynomial (contd)

$$y(x)=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)\left(\frac{x-x_4}{x_0-x_4}\right)\left(\frac{x-x_5}{x_0-x_5}\right)\left(\frac{x-x_6}{x_0-x_6}\right)y(x_0)$$

$$+\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)\left(\frac{x-x_4}{x_1-x_4}\right)\left(\frac{x-x_5}{x_1-x_5}\right)\left(\frac{x-x_6}{x_1-x_6}\right)y(x_1)$$

$$+\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)\left(\frac{x-x_6}{x_2-x_6}\right)y(x_2)$$

$$+\left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)\left(\frac{x-x_6}{x_3-x_6}\right)y(x_3)$$

$$+\left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)\left(\frac{x-x_6}{x_4-x_6}\right)y(x_4)$$

$$+\left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)\left(\frac{x-x_6}{x_5-x_6}\right)y(x_5)$$

$$+\left(\frac{x-x_0}{x_6-x_0}\right)\left(\frac{x-x_1}{x_6-x_1}\right)\left(\frac{x-x_2}{x_6-x_2}\right)\left(\frac{x-x_3}{x_6-x_3}\right)\left(\frac{x-x_4}{x_6-x_4}\right)\left(\frac{x-x_5}{x_6-x_5}\right)y(x_6)$$

Sixth Order Polynomial (contd)

$$y(x)=\frac{(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(2.20-1.28)(2.20-0.66)(2.20-0.00)(2.20+0.60)(2.20+1.04)(2.20+1.20)}(0.00)$$

$$+\frac{(x-2.20)(x-0.66)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(1.28-2.20)(1.28-0.66)(1.28-0.00)(1.28+0.60)(1.28+1.04)(1.28+1.20)}(0.88)$$

$$+\frac{(x-2.20)(x-1.28)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(0.66-2.20)(0.66-1.28)(0.66-0.00)(0.66+0.60)(0.66+1.04)(0.66+1.20)}(1.14)$$

$$+\frac{(x-2.20)(x-1.28)(x-0.66)(x+0.60)(x+1.04)(x+1.20)}{(0.00-2.20)(0.00-1.28)(0.00-0.66)(0.00+0.60)(0.00+1.04)(0.00+1.20)}(1.20)$$

$$+\frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+1.04)(x+1.20)}{(-.60-2.20)(-.60-1.28)(-.60-0.66)(-.60-0.00)(-.60+1.04)(-.60+1.20)}(1.04)$$

$$+\frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.20)}{(-1.04-2.20)(-1.04-1.28)(-1.04-0.66)(-1.04-0.00)(-1.04+0.60)(-1.04+1.20)}(0.60)$$

$$+\frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.04)}{(-1.20-2.20)(-1.20-1.28)(-1.20-0.66)(-1.20-0.00)(-1.20+0.60)(-1.20+1.04)}(0.00)$$

Sixth Order Polynomial (contd)

$$= \frac{x^6 - 0.02x^5 - 4.0784x^4 - 2.54064x^3 + 1.62202x^2 + 1.08726x}{-8.97438}$$

$$+ \frac{x^6 - 0.64x^5 - 4.4752x^4 - 0.27392x^3 + 4.69325x^2 + 2.10862x}{2.20234}$$

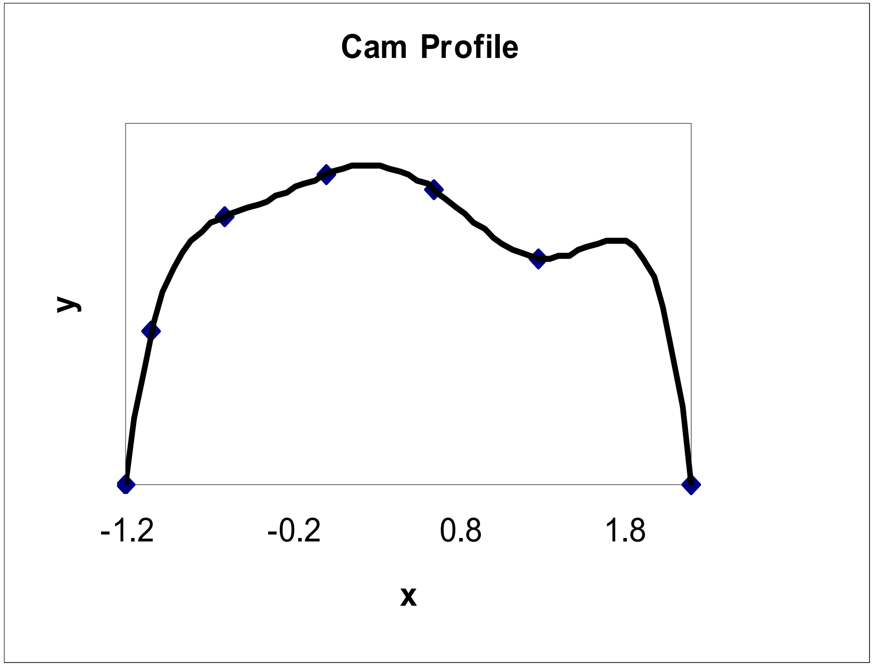
$$+ \frac{x^6 - 1.3x^5 - 4.0528x^4 + 2.67971x^3 + 4.87404x^2 - 0.988923x - 1.39169}{-1.15974}$$

$$+ \frac{x^6 - 1.9x^5 - 2.9128x^4 + 4.42739x^3 + 2.2176x^2 - 2.31948x}{1.01020}$$

$$+ \frac{x^6 - 2.34x^5 - 1.6192x^4 + 4.36368x^3 + 0.335808x^2 - 1.33816x}{-1.55933}$$

$$y(x) = 1.2 + 0.25111x - 0.27256x^2 - 0.56764x^3$$
$$+ 0.07201x^4 + 0.45240x^5 - 0.17103x^6, \quad -1.20 \le x \le 2.20$$

Sixth Order Polynomial (contd)



$$y(x) = 1.2 + 0.25111x - 0.27256x^2 - 0.56764x^3 + 0.07201x^4 + 0.45240x^5 - 0.17103x^6, \quad -1.20 \le x \le 2.20$$