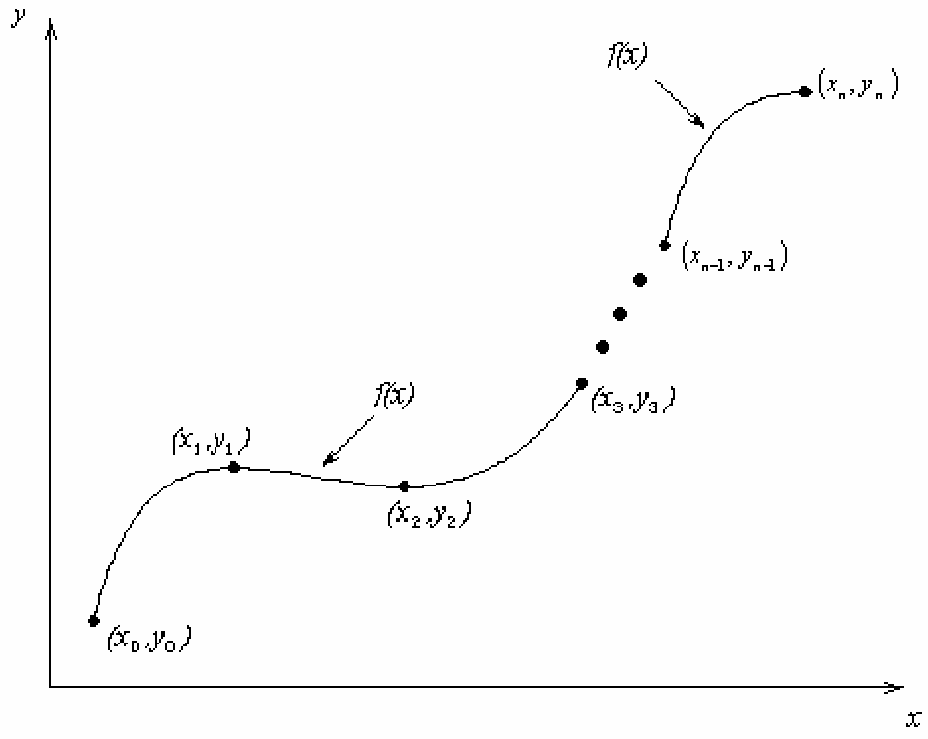
Interpolation

Topic: Newton's Divided Difference Polynomial Method

Major: Industrial

What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

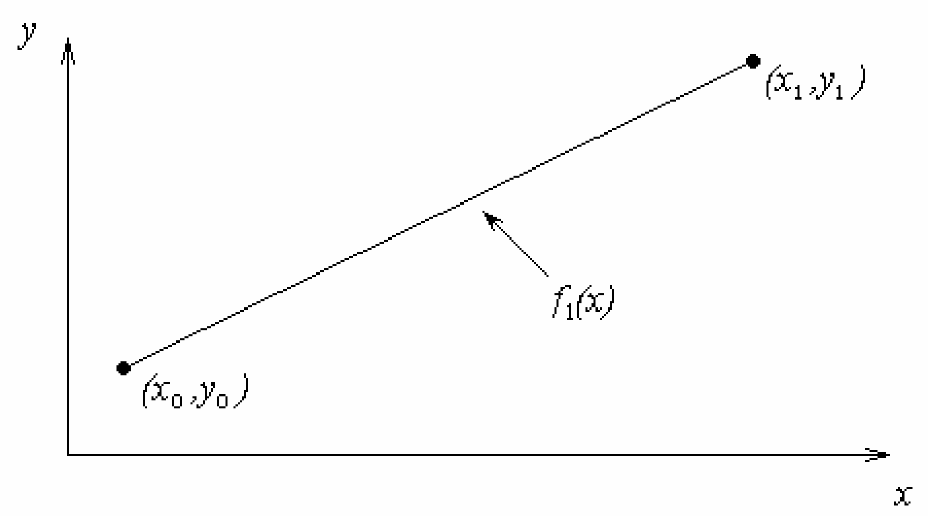
<u>Linear interpolation</u>: Given $(x_0, y_0), (x_1, y_1)$, pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$

where

$$b_0 = f(x_0)$$

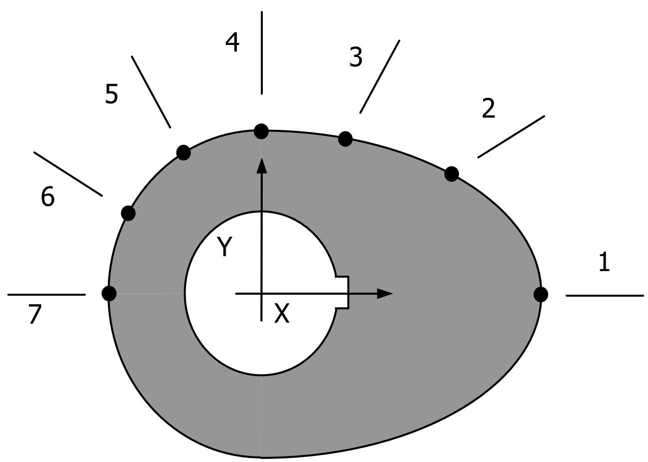
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

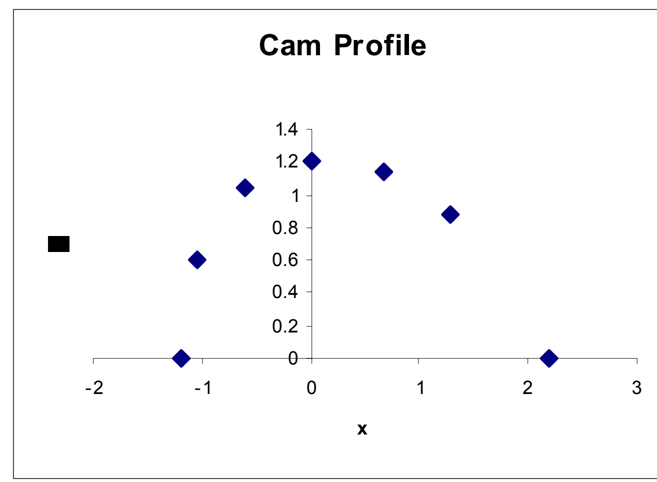


Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the Newton Divided Difference method.

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0





Linear Interpolation

$$y(x) = b_0 + b_1(x - x_0)$$

$$x_0 = 1.28,\ y(x_0) = 0.88$$

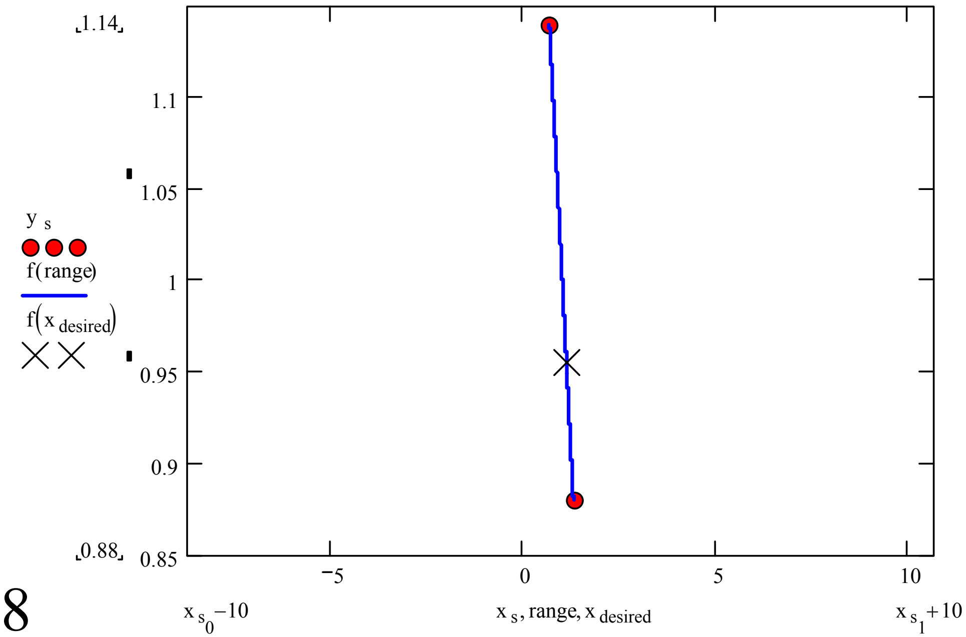
$$x_1 = 0.66,\ y(x_1) = 1.14$$

$$b_0 = y(x_0)$$

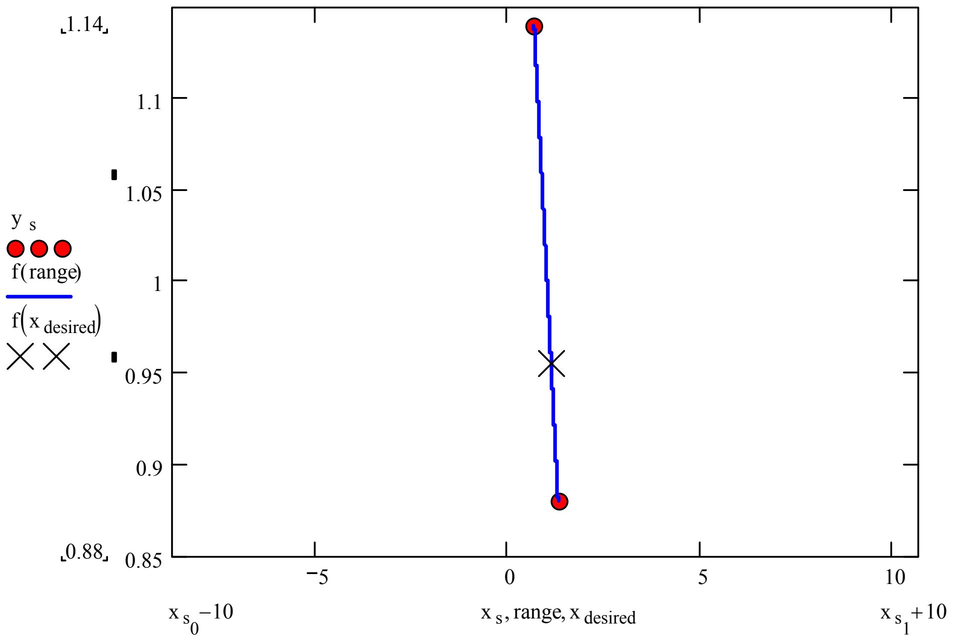
$$= 0.88$$

$$b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{1.14 - 0.88}{0.66 - 1.28}$$

$$= -0.41935$$



Linear Interpolation (contd)



$$y(x) = b_0 + b_1(x - x_0)$$

$$= 0.88 - 0.41935(x - 1.28), \qquad 0.66 \le x \le 1.28$$

At $x = 1.10$

$$y(4.00) = 0.88 - 0.41935(1.10 - 1.28)$$

$$= 0.95548 \ in.$$

Quadratic Interpolation

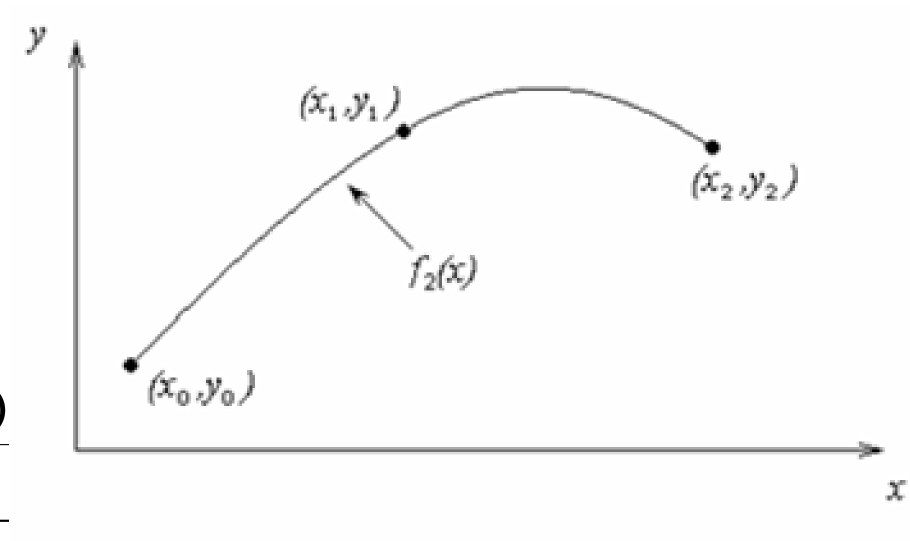
Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

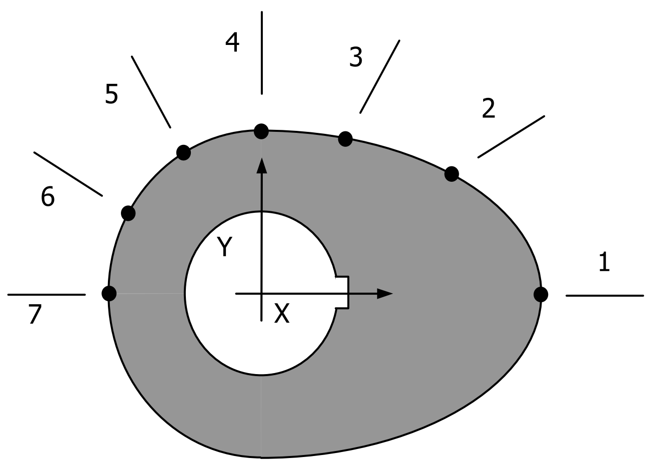
$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

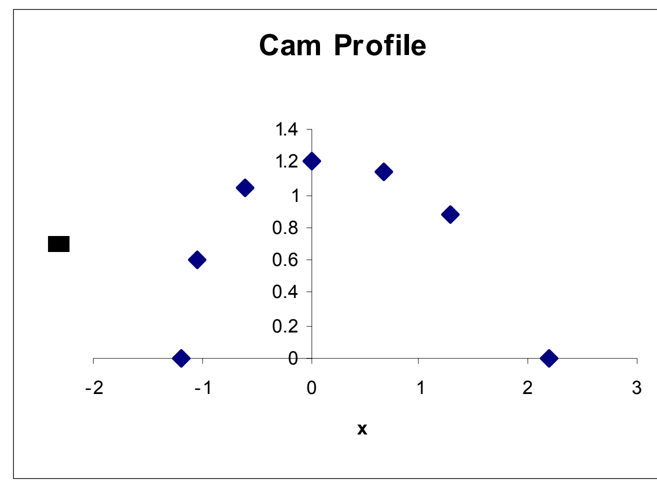


Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a quadratic profile from x=2.2 to x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the Newton Divided Difference method.

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0





Quadratic Interpolation (contd)

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$x_0 = 2.20, \; y(x_0) = 0.00$$

$$x_1 = 1.28, \; y(x_1) = 0.88$$

$$x_2 = 0.66, \; y(x_2) = 1.14$$

Quadratic Interpolation (contd)

$$b_0 = y(x_0)$$

$$= 0.00$$

$$b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{0.88 - 0.00}{1.28 - 2.20}$$

$$= -0.95652$$

$$b_2 = \frac{\frac{y(x_2) - y(x_1)}{x_2 - x_1} - \frac{y(x_1) - y(x_0)}{x_1 - x_0}}{x_2 - x_0} = \frac{\frac{1.14 - 0.88}{0.66 - 1.28} - \frac{0.88 - 0.00}{1.28 - 2.20}}{0.66 - 2.20}$$

$$= \frac{-0.41935 + 0.95652}{-1.54}$$

$$= -0.34881$$

Quadratic Interpolation (contd)

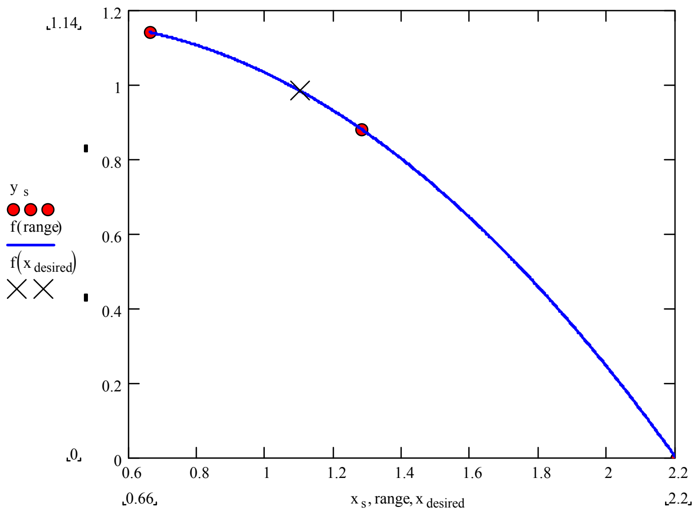
$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= 0 - 0.95652(x - 2.20) - 0.34881(x - 2.20)(x - 1.28), \quad 0.66 \le x \le 2.20$$

At $x = 1.10$,

$$y(1.10) = 0 - 0.95652(1.10 - 2.20) - 0.34881(1.10 - 2.20)(1.10 - 1.28)$$

$$= 0.98311 \ in.$$



Quadratic Interpolation (contd)

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{0.98311 - 0.95548}{0.98311}\right| \times 100$$

$$= 2.810\ \%$$

General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General Form

Given $(n+1)$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

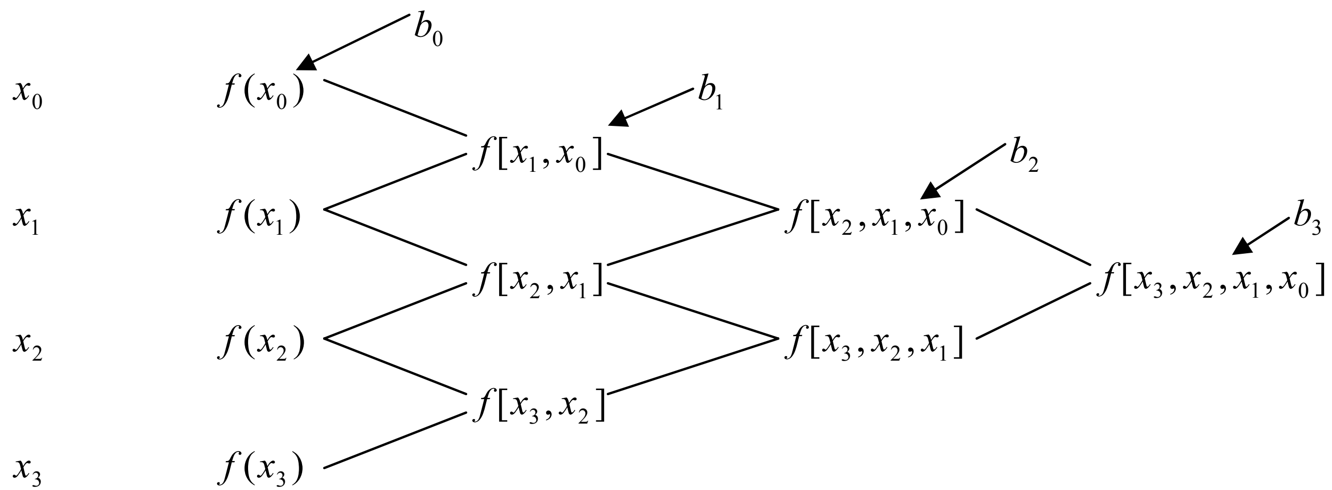
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots., x_0]$$

$$b_n = f[x_n, x_{n-1}, \ldots., x_0]$$

General form

The third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3), is

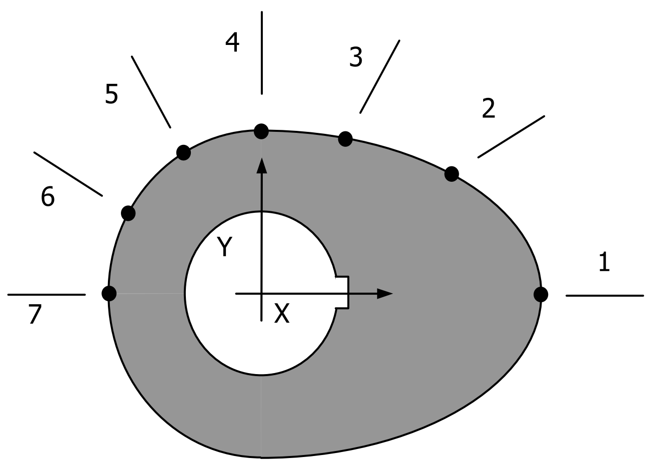
$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

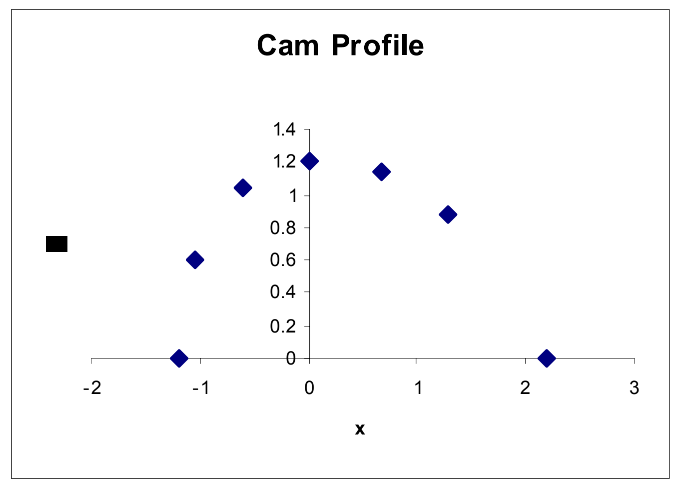


Example

A curve needs to be fit through the seven points to fabricate the cam. Find the cam profile using all the seven points using Newton's divided difference method for sixth order polynomial interpolation

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0





Example

The value of y profile is chosen as

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$
$$+ b_6(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$$

$x_0 = 2.20, \quad y(x_0) = 0.00$

$x_1 = 1.28, \quad y(x_1) = 0.88$

$x_2 = 0.66, \quad y(x_2) = 1.14$

$x_3 = 0.00, \quad y(x_3) = 1.20$

$x_4 = -.60, \quad y(x_4) = 1.04$

$x_5 = -1.04, \quad y(x_5) = 0.60$

$x_6 = -1.20, \quad y(x_6) = 0.00$

Example

The values of the constants are found as:

$$b_0 = 0.00$$

$$b_1 = -0.95652$$

$$b_2 = -0.34881$$

$$b_3 = -0.041916$$

$$b_4 = -0.020137$$

$$b_5 = 0.024833$$

$$b_6 = -0.17103$$

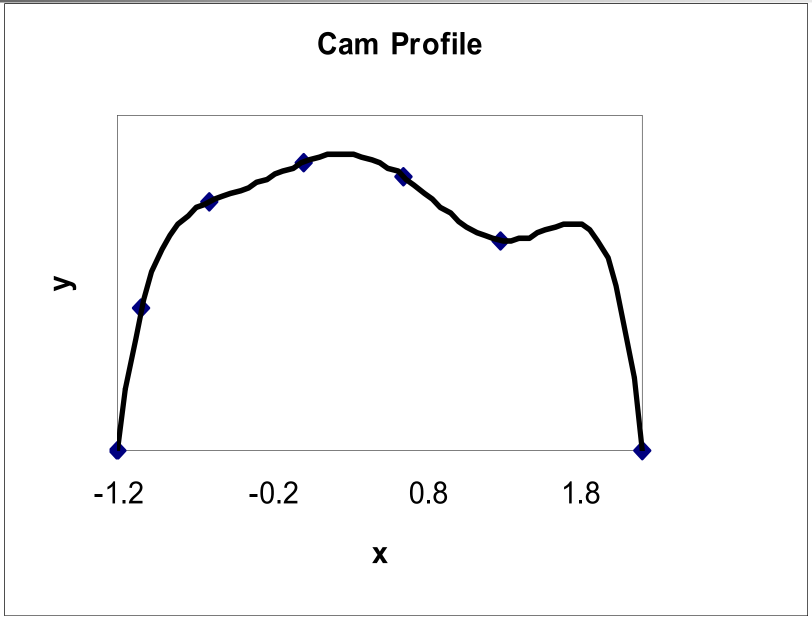
Example

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$
$$+ b_6(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$$

$$= 0 - .95652(x - 2.2) - .34881(x - 2.2)(x - 1.28) - .041916(x - 2.2)(x - 1.28)(x - 0.66)$$
$$- .020137(x - 2.2)(x - 1.28)(x - 0.66)(x - 0)$$
$$+ .024833(x - 2.2)(x - 1.28)(x - 0.66)(x - 0)(x + 0.6)$$
$$- .17103(x - 2.2)(x - 1.28)(x - 0.66)(x - 0)(x + 0.6)(x + 1.04)$$

$$y(x) = 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3$$
$$+ 0.072013x^4 + 0.45241x^5 - 0.17103x^6, \quad -1.20 \le x \le 2.20$$

Example



$$y(x) = 0 - .95652(x-2.2) - .34881(x-2.2)(x-1.28) - .041916(x-2.2)(x-1.28)(x-0.66)$$
$$- .020137(x-2.2)(x-1.28)(x-0.66)(x-0)$$
$$+ .024833(x-2.2)(x-1.28)(x-0.66)(x-0)(x+0.6)$$
$$- .17103(x-2.2)(x-1.28)(x-0.66)(x-0)(x+0.6)(x+1.04)$$