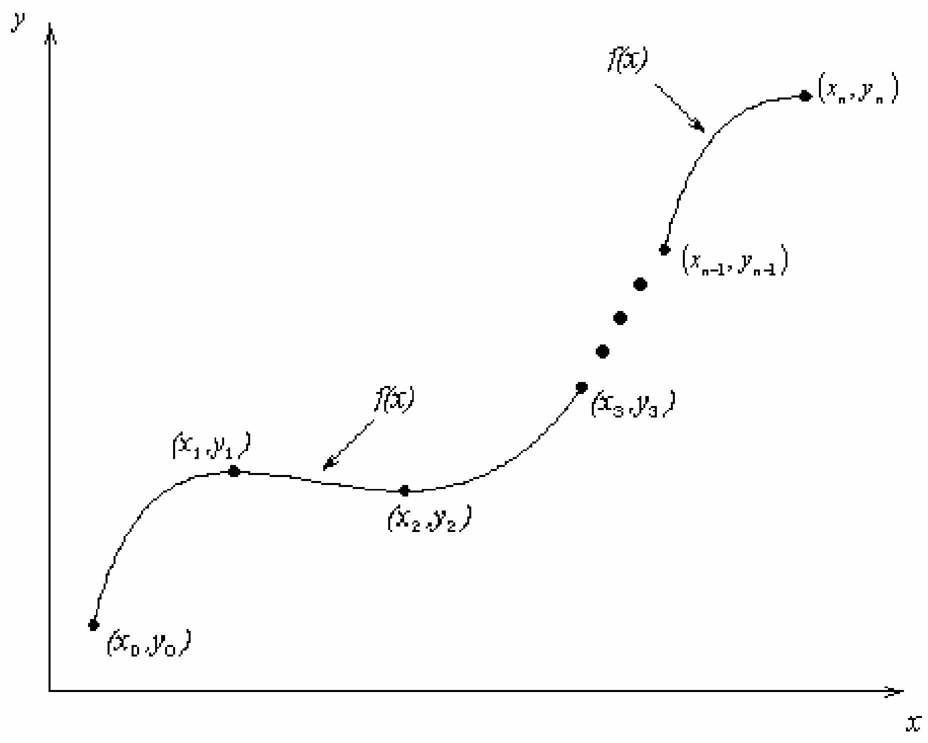
Interpolation

Topic: Spline Interpolation Method

Major: Industrial

What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

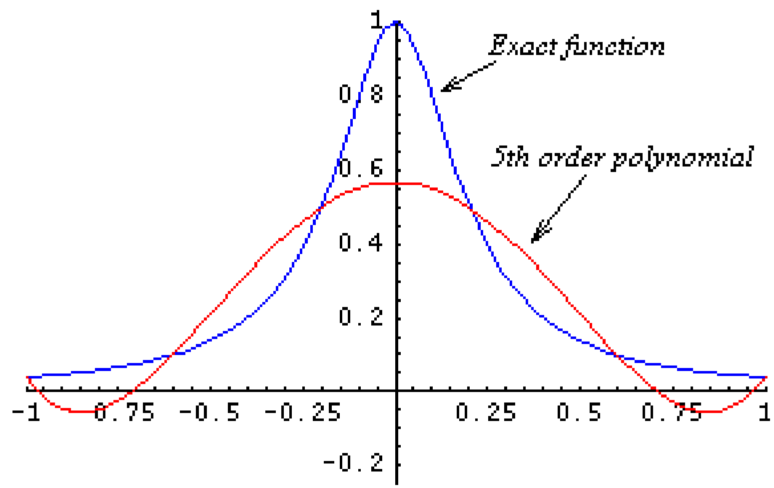


Figure : 5th order polynomial vs. exact function

Why Splines ?

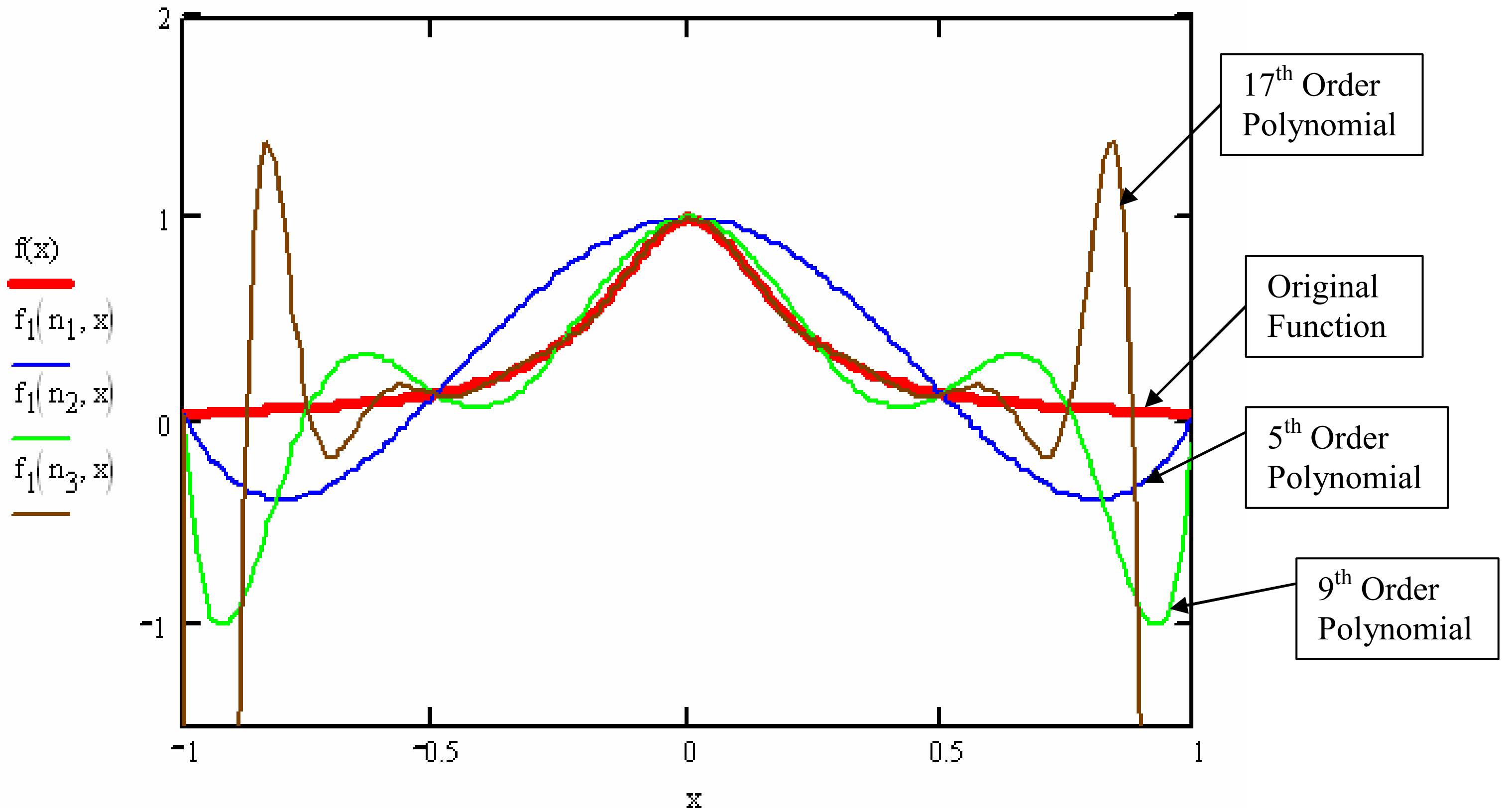
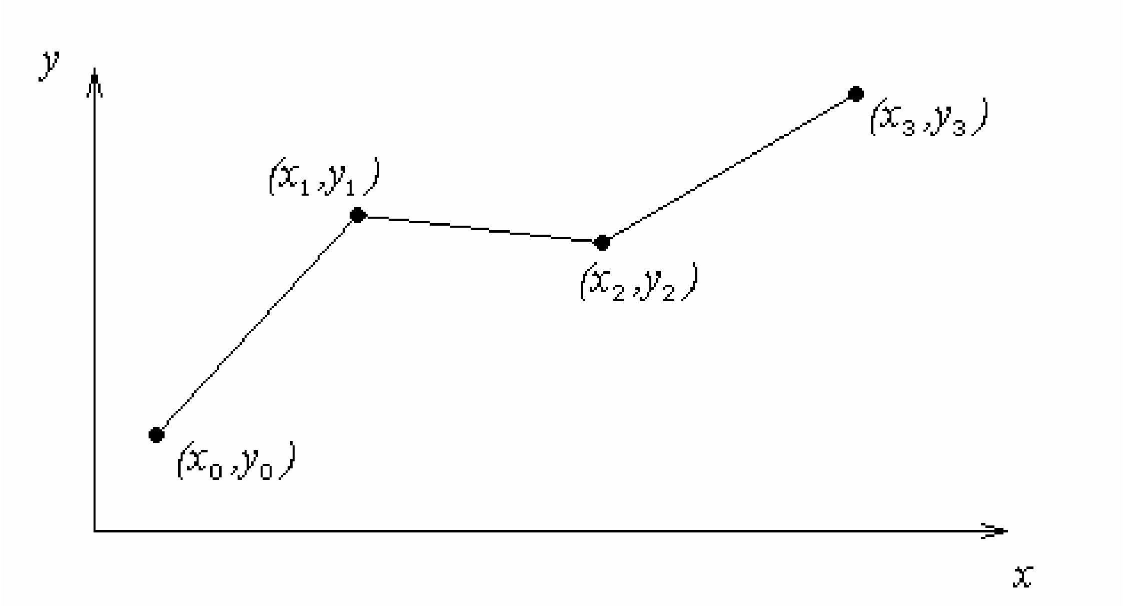


Figure : Higher order polynomial interpolation is a bad idea

Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $(y_i = f(x_i))$

Figure : Linear splines



Linear Interpolation (contd)

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2$$

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$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n$$

Note the terms of

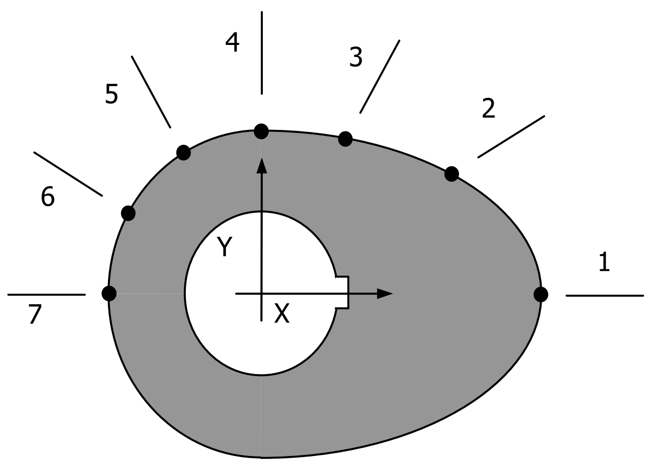
$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

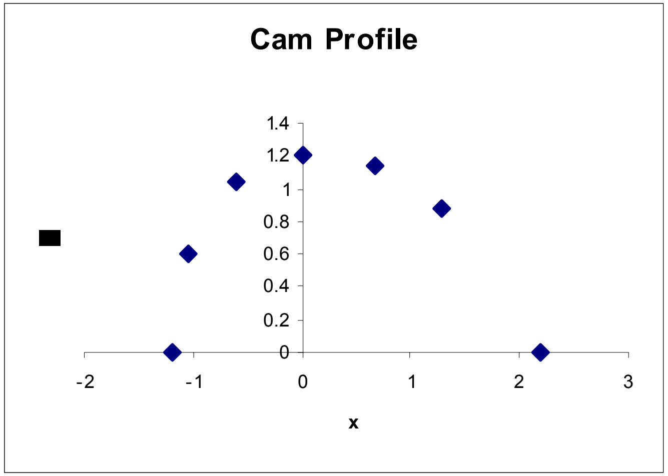
in the above function are simply slopes between x_{i-1} and x_i.

Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between x = 1.28 to x = 0.66, find the value of y at x=1.1 ? Find using the Spline Interpolation method.

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0





Linear Interpolation

$$x_0 = 1.28, \qquad y(x_0) = 0.88$$

$$x_1 = 0.66, \qquad y(x_1) = 1.14$$

$$y(x) = y(x_0) + \frac{y(x_1) - y(x_0)}{x_1 - x_0}(x - x_0)$$

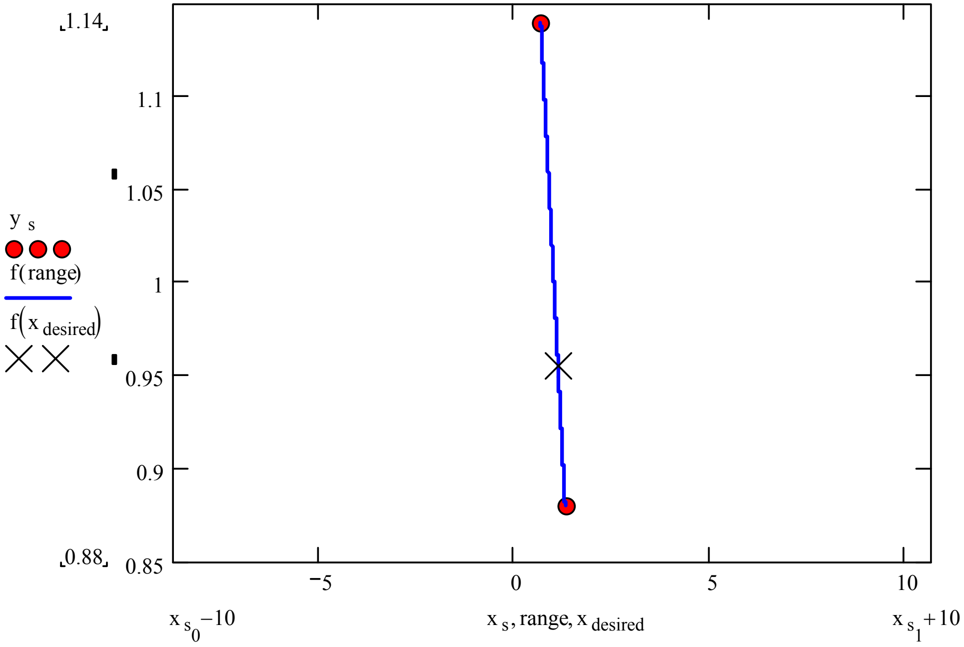
$$= 0.88 + \frac{1.14 - 0.88}{0.66 - 1.28}(x - 1.28)$$

$$y(x) = 0.88 - 0.41935(x - 1.28)$$

At $x = 1.10,$

$$y(1.10) = 0.88 - 0.41935(1.10 - 1.28)$$

$$= 0.95548 \; in.$$



Linear Interpolation

Find the cam profile using linear splines.

The first linear spline connects $x = -1.20$ and $x = -1.04$:

$$y(x) = 0.00 + \frac{0.60 - 0.00}{-1.04 + 1.20}(x + 1.20)$$

$$y(x) = 3.75(x + 1.20) \qquad -1.20 \le x \le -1.04$$

Similarly,

$$y(x) = x + 1.64, \qquad -1.04 \le x \le -0.60$$

$$y(x) = 1.04 + 0.26667(x + 0.60), \qquad -0.60 \le x \le 0.00$$

$$y(x) = 1.20 - 0.090909x, \qquad 0.00 \le x \le 0.66$$

$$y(x) = 1.14 - 0.41935(x - 0.66), \qquad 0.66 \le x \le 1.28$$

$$y(x) = 0.88 - 0.95652(x - 1.28), \qquad 1.28 \le x \le 2.20$$

Linear Interpolation

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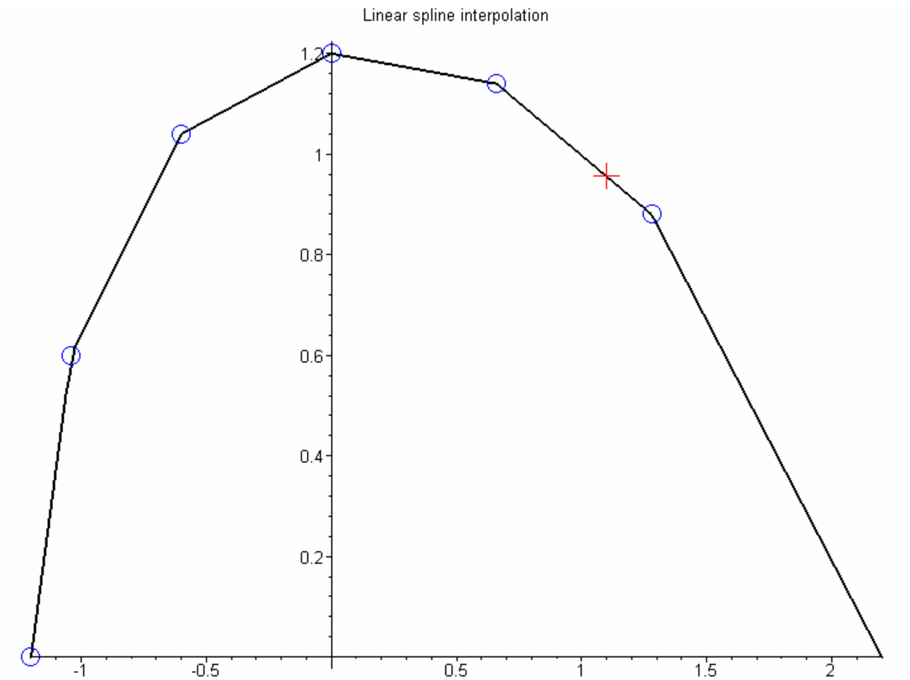

Quadratic Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$f(x) = a_1x^2 + b_1x + c_1, \qquad x_0 \le x \le x_1$$

$$= a_2x^2 + b_2x + c_2, \qquad x_1 \le x \le x_2$$

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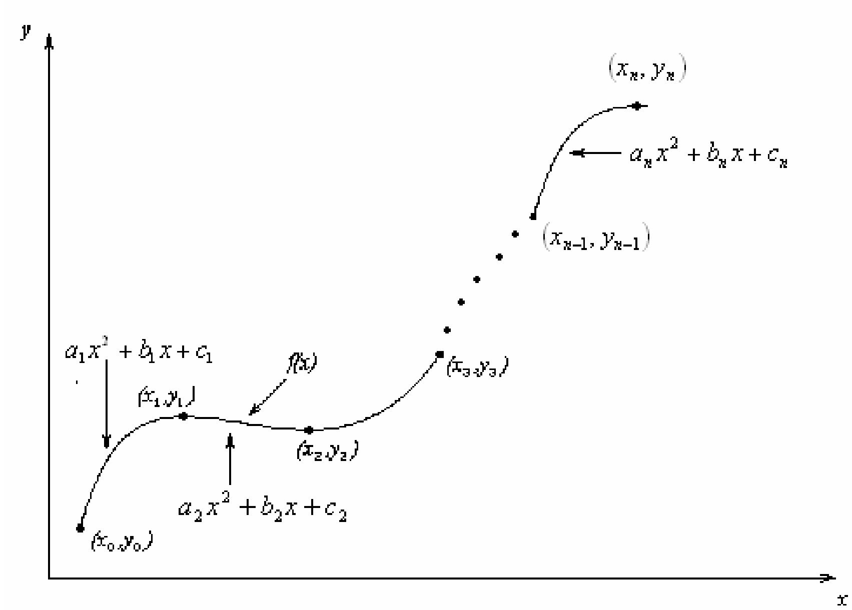
$$= a_nx^2 + b_nx + c_n, \qquad x_{n-1} \le x \le x_n$$



Find $a_i, b_i, c_i, i = 1, 2, \ldots, n$

Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^{\ 2} + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^{\ 2} + b_1 x_1 + c_1 = f(x_1)$$

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$$a_i x_{i-1}^{\ 2} + b_i x_{i-1} + c_i = f(x_{i-1})$$

$$a_i x_i^{\ 2} + b_i x_i + c_i = f(x_i)$$

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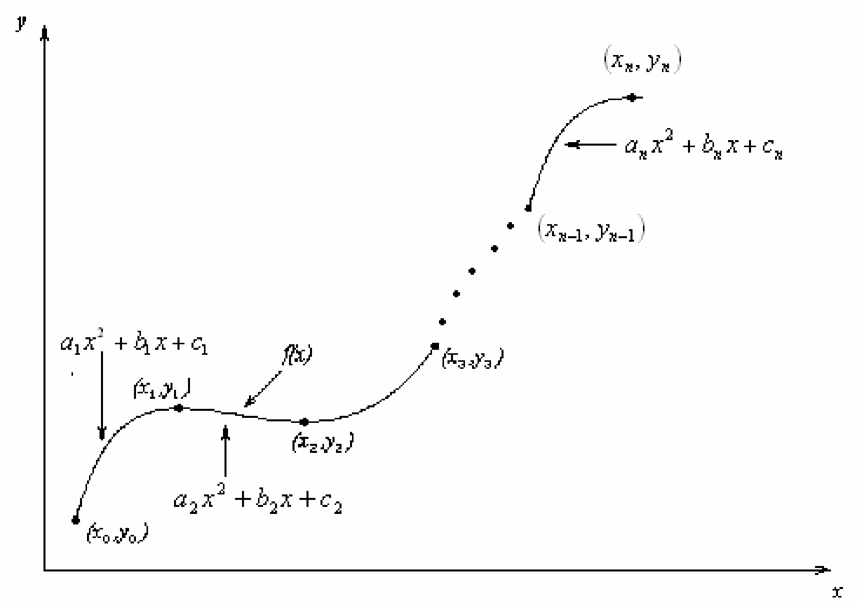
$$a_n x_{n-1}^{\ 2} + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^{\ 2} + b_n x_n + c_n = f(x_n)$$



This condition gives 2n equations

Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \quad \text{is} \quad 2a_1x + b_1$$

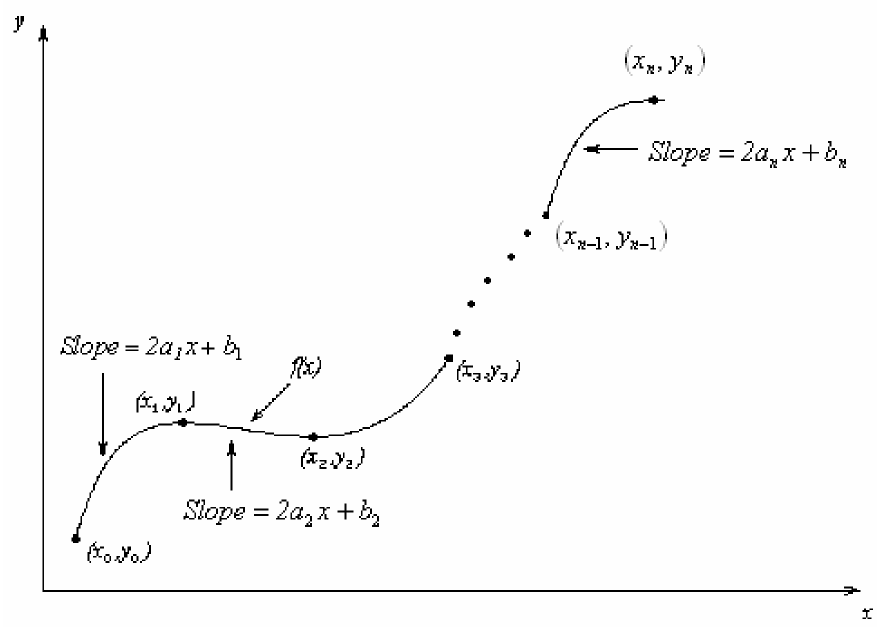
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \quad \text{is} \quad 2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



Quadratic Splines (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

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$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

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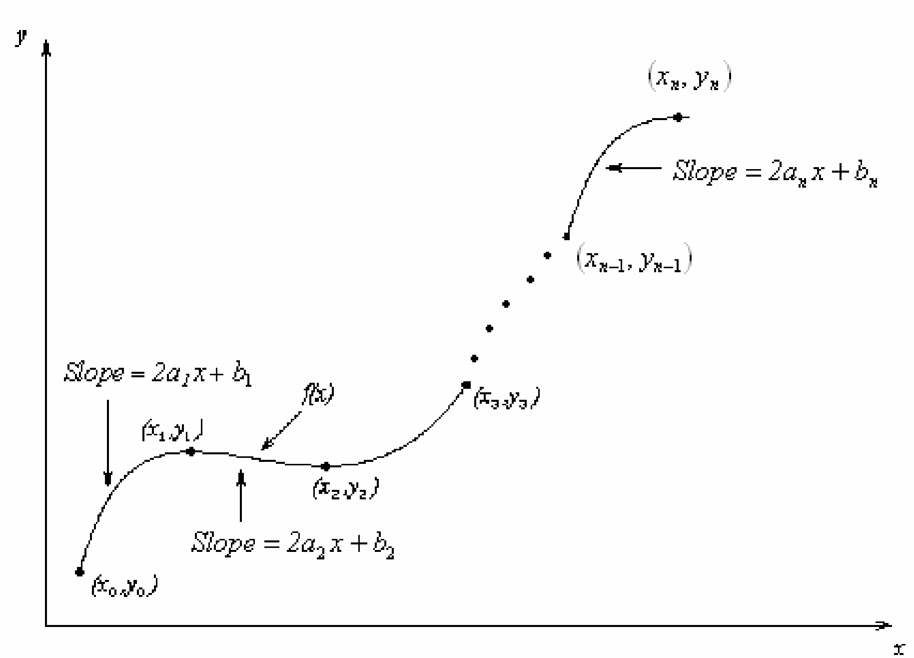
$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



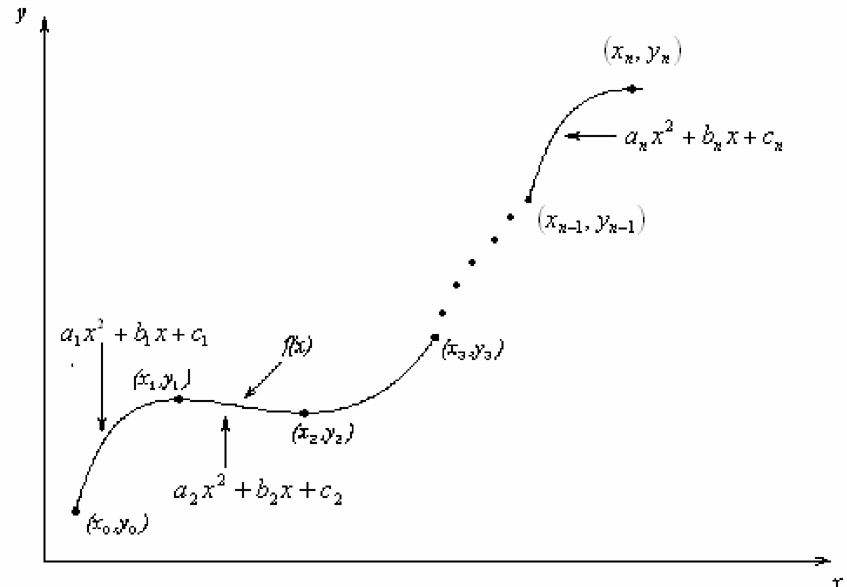
We have (n-1) such equations. The total number of equations is $(2n) + (n-1) = (3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

Quadratic Splines (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

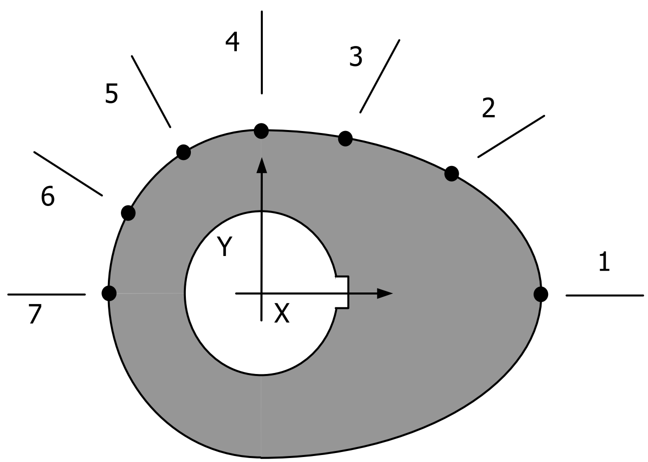
$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, \qquad & x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, \qquad & x_1 \le x \le x_2 \\
&\;\;\vdots \\
&= a_n x^2 + b_n x + c_n, \qquad & x_{n-1} \le x \le x_n
\end{aligned}
$$

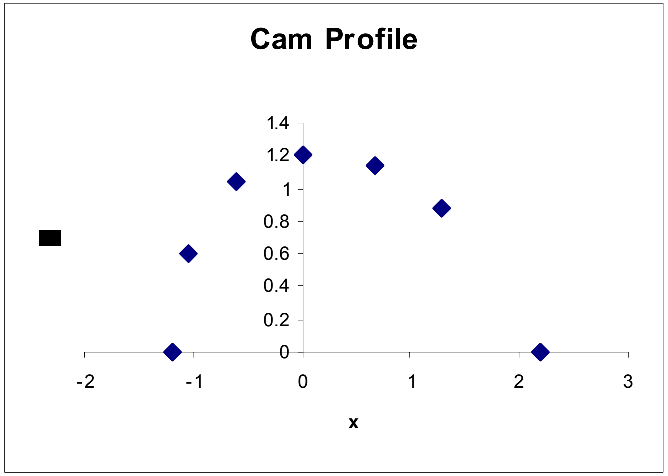


Example

A curve needs to be fit through the given points to fabricate the cam. Find the cam profile using quadratic splines.

Point	x	y
1	2.2	0
2	1.28	0.88
3	0.66	1.14
4	0	1.2
5	-0.6	1.04
6	-1.04	0.6
7	-1.2	0



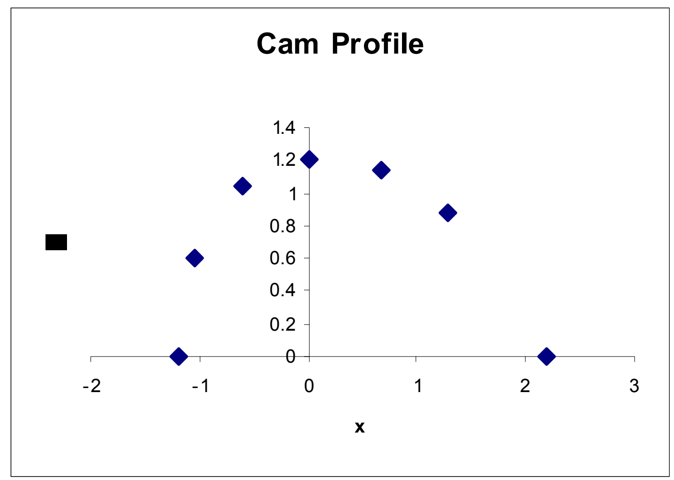


Solution

Since there are seven data points,

six quadratic splines pass through them.

$$
\begin{aligned}
y(x) &= a_1x^2 + b_1x + c_1, & -1.20 \le x \le -1.04 \\
&= a_2x^2 + b_2x + c_2, & -1.04 \le x \le -0.60 \\
&= a_3x^2 + b_3x + c_3, & -0.60 \le x \le 0.00 \\
&= a_4x^2 + b_4x + c_4, & 0.00 \le x \le 0.66 \\
&= a_5x^2 + b_5x + c_5, & 0.66 \le x \le 1.28 \\
&= a_6x^2 + b_6x + c_6, & 1.28 \le x \le 2.20
\end{aligned}
$$



Solution (contd)

Setting up the equations

Each quadratic spline passes through two consecutive data points giving

$a_1x^2 + b_1x + c_1$ passes through x = -1.20 and x = -1.04,

$$a_1(-1.20)^2 + b_1(-1.20) + c_1 = 0.00 \quad (1)$$

$$a_1(-1.04)^2 + b_1(-1.04) + c_1 = 0.60 \quad (2)$$

Similarly,

$$a_2(-1.04)^2 + b_2(-1.04) + c_2 = 0.60 \quad (3)$$

$$a_2(-0.60)^2 + b_2(-0.60) + c_2 = 1.04 \quad (4)$$

$$a_3(-0.60)^2 + b_3(-0.60) + c_3 = 1.04 \quad (5)$$

$$a_3(0.00)^2 + b_3(0.00) + c_3 = 1.20 \quad (6)$$

$$a_4(0.00)^2 + b_4(0.00) + c_4 = 1.20 \quad (7)$$

$$a_4(0.66)^2 + b_4(0.66) + c_4 = 1.14 \quad (8)$$

$$a_5(0.66)^2 + b_5(0.66) + c_5 = 1.14 \quad (9)$$

$$a_5(1.28)^2 + b_5(1.28) + c_5 = 0.88 \quad (10)$$

$$a_6(1.28)^2 + b_6(1.28) + c_6 = 0.88 \quad (11)$$

$$a_6(2.20)^2 + b_6(2.20) + c_6 = 0.00 \quad (12)$$

Solution (contd)

Quadratic splines have continuous derivatives at the interior data points

At x = -1.04

$$2a_1(-1.04) + b_1 - 2a_2(-1.04) - b_2 = 0 \qquad (13)$$

At x = -0.60

$$2a_2(-0.60) + b_2 - 2a_3(-0.60) - b_3 = 0 \qquad (14)$$

At x = 0.00

$$2a_3(0.00) + b_3 - 2a_4(0.00) - b_4 = 0 \qquad (15)$$

At x = 0.66

$$2a_4(0.66) + b_4 - 2a_5(0.66) - b_5 = 0 \qquad (16)$$

At x = 1.28

$$2a_5(1.28) + b_5 - 2a_6(1.28) - b_6 = 0 \qquad (17)$$

Assuming the first spline $a_1x^2 + b_1x + c_1$ is linear,

$$a_1 = 0 \qquad (18)$$

Solution (contd)

$$
\begin{bmatrix}
1.44 & -1.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.0816 & -1.04 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.0816 & -1.04 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.36 & -0.6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.36 & -0.6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4356 & 0.66 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4356 & 0.66 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6384 & 1.28 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6384 & 1.28 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.84 & 2.2 & 1 \\
-2.08 & 1 & 0 & 2.08 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.2 & 1 & 0 & 1.2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.32 & 1 & 0 & -1.32 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.56 & 1 & 0 & -2.56 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \\ a_6 \\ b_6 \\ c_6 \end{bmatrix}
=
\begin{bmatrix} 0.00 \\ 0.60 \\ 0.60 \\ 1.04 \\ 1.04 \\ 1.20 \\ 1.20 \\ 1.14 \\ 1.14 \\ 0.88 \\ 0.88 \\ 0.00 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Solution (contd)

Solving the above 18 equations gives the 18 unknowns as

i	a_i	b_i	c_i
1	0	3.75	4.5
2	-6.25	-9.25	-2.26
3	3.3611	2.2833	1.2
4	-3.5973	2.2833	1.2
5	3.2997	-6.8207	4.2043
6	-2.8076	8.8138	-5.8018

Solution (contd)

Therefore, the splines are given by

$$
\begin{aligned}
y(x) &= 3.75x + 4.5, & -1.20 \le x \le -1.04 \\
&= -6.25x^2 - 9.25x - 2.26, & -1.04 \le x \le -0.60 \\
&= 3.3611x^2 + 2.2833x + 1.2, & -0.60 \le x \le 0.00 \\
&= -3.5973x^2 + 2.2833x + 1.2, & 0.00 \le x \le 0.66 \\
&= 3.2997x^2 - 6.8207x + 4.2043, & 0.66 \le x \le 1.28 \\
&= -2.8076x^2 + 8.8138x - 5.8018, & 1.28 \le x \le 2.20
\end{aligned}
$$

Comparison

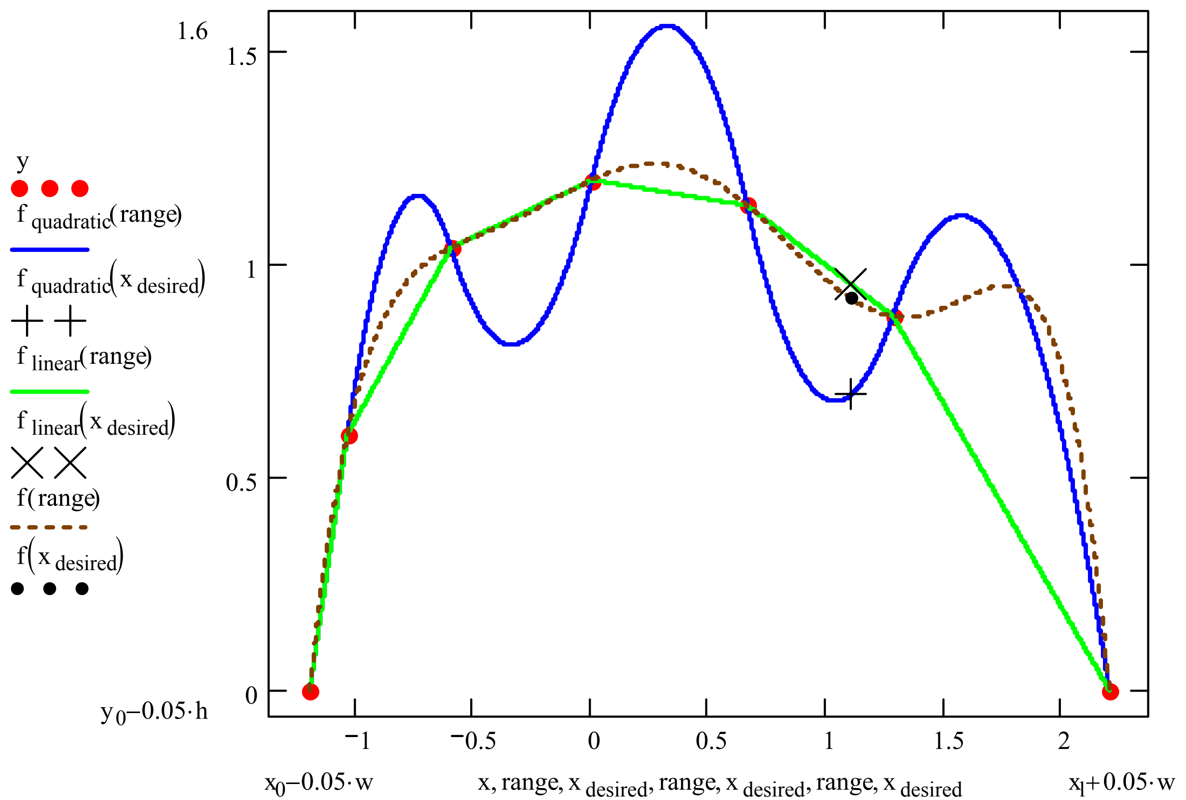


Figure : Cam profile as defined by linear and quadratic splines

(dotted line represents sixth order polynomial interpolant)