Integration

Topic: Gauss Quadrature Rule of Integration

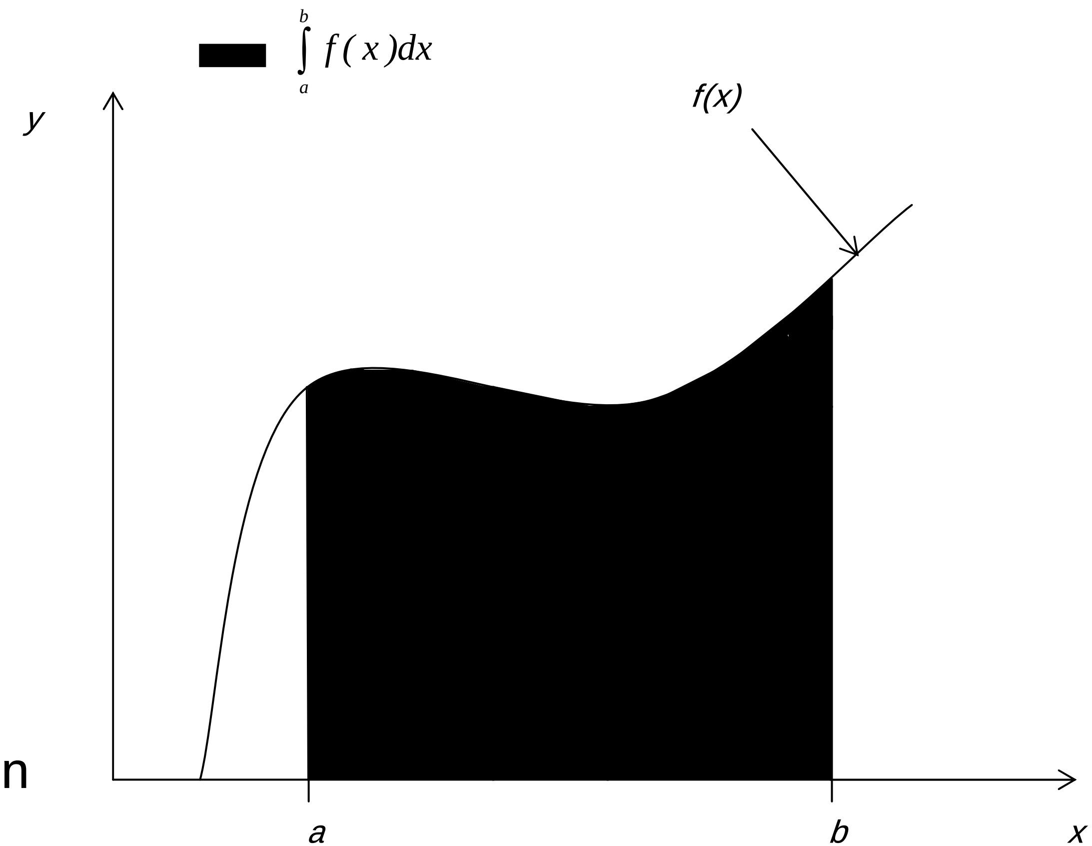
Major: Industrial Engineering

What is Integration?

Integration

The process of measuring the area under a curve.

$$I = \int_a^b f(x)dx$$



Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration

Two-Point Gaussian Quadrature Rule

Basis of the Gaussian Quadrature Rule

Previously, the Trapezoidal Rule was developed by the method of undetermined coefficients. The result of that development is summarized below.

$$\int_a^b f(x)dx \cong c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

Basis of the Gaussian Quadrature Rule

The two-point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the arguments of the function are not predetermined as a and b but as unknowns x_1 and x_2. In the two-point Gauss Quadrature Rule, the integral is approximated as

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

Basis of the Gaussian Quadrature Rule

The four unknowns x_1, x_2, c_1 and c_2 are found by assuming that the formula gives exact results for integrating a general third order polynomial, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

Hence

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

Basis of the Gaussian Quadrature Rule

It follows that

$$\int_a^b f(x)dx = c_1\left(a_0 + a_1x_1 + a_2x_1^{\ 2} + a_3x_1^{\ 3}\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^{\ 2} + a_3x_2^{\ 3}\right)$$

Equating Equations the two previous two expressions yield

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

$$= c_1\left(a_0 + a_1x_1 + a_2x_1^{\ 2} + a_3x_1^{\ 3}\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^{\ 2} + a_3x_2^{\ 3}\right)$$

$$= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2\left(c_1x_1^{\ 2} + c_2x_2^{\ 2}\right) + a_3\left(c_1x_1^{\ 3} + c_2x_2^{\ 3}\right)$$

Basis of the Gaussian Quadrature Rule

Since the constants a_0, a_1, a_2, a_3 are arbitrary

$$b-a=c_1+c_2$$

$$\frac{b^2-a^2}{2}=c_1x_1+c_2x_2$$

$$\frac{b^3-a^3}{3}=c_1x_1{}^2+c_2x_2{}^2$$

$$\frac{b^4-a^4}{4}=c_1x_1{}^3+c_2x_2{}^3$$

Basis of Gauss Quadrature

The previous four simultaneous nonlinear Equations have only one acceptable solution,

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

Basis of Gauss Quadrature

Hence Two-Point Gaussian Quadrature Rule

$$\int_a^b f(x)dx \approx$$

$$c_1 f(x_1) + c_2 f(x_2) \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right)+\frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right)$$

Higher Point Gaussian Quadrature Formulas

Higher Point Gaussian Quadrature Formulas

$$\int_a^b f(x)dx \cong c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

is called the three-point Gauss Quadrature Rule.

The coefficients c_1, c_2, and c_3, and the functional arguments x_1, x_2, and x_3 are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right) dx$$

General n-point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots\ldots + c_n f(x_n)$$

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

In handbooks, coefficients and arguments given for n-point Gauss Quadrature Rule are given for integrals

$$\int_{-1}^{1} g(x)dx \cong \sum_{i=1}^{n} c_i g(x_i)$$

as shown in Table 1.

Table 1: Weighting factors c and function arguments x used in Gauss Quadrature Formulas.

Points	Weighting Factors	Function Arguments
2	c_1 = 1.000000000 c_2 = 1.000000000	x_1 = -0.577350269 x_2 = 0.577350269
3	c_1 = 0.555555556 c_2 = 0.888888889 c_3 = 0.555555556	x_1 = -0.774596669 x_2 = 0.000000000 x_3 = 0.774596669
4	c_1 = 0.347854845 c_2 = 0.652145155 c_3 = 0.652145155 c_4 = 0.347854845	x_1 = -0.861136312 x_2 = -0.339981044 x_3 = 0.339981044 x_4 = 0.861136312

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

Table 1 (cont.) : Weighting factors c and function arguments x used in Gauss Quadrature Formulas.

Points	Weighting Factors	Function Arguments
5	c_1 = 0.236926885 c_2 = 0.478628670 c_3 = 0.568888889 c_4 = 0.478628670 c_5 = 0.236926885	x_1 = -0.906179846 x_2 = -0.538469310 x_3 = 0.000000000 x_4 = 0.538469310 x_5 = 0.906179846
6	c_1 = 0.171324492 c_2 = 0.360761573 c_3 = 0.467913935 c_4 = 0.467913935 c_5 = 0.360761573 c_6 = 0.171324492	x_1 = -0.932469514 x_2 = -0.661209386 x_3 = -0.238619186 x_4 = 0.238619186 x_5 = 0.661209386 x_6 = 0.932469514

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

So if the table is given for $\int_{-1}^{1} g(x)dx$ integrals, how does one solve $\int_{a}^{b} f(x)dx$? The answer lies in that any integral with limits of $[a, b]$ can be converted into an integral with limits $[-1, 1]$ Let

$$x = mt + c$$

If $x = a,$ then $t = -1$

If $x = b,$ then $t = 1$

Such that:

$$m = \frac{b-a}{2}$$

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

Then $$c = \frac{b+a}{2}$$ Hence

$$x = \frac{b-a}{2}t + \frac{b+a}{2} \qquad dx = \frac{b-a}{2}dt$$

Substituting our values of x, and dx into the integral gives us

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2}dx$$

Example 1

For an integral $\int_a^b f(x)dx,$ derive the one-point Gaussian Quadrature Rule.

Solution

The one-point Gaussian Quadrature Rule is

$$\int_a^b f(x)dx \approx c_1 f(x_1)$$

Solution

Assuming the formula gives exact values for integrals

$$\int_{-1}^{1} 1dx, \quad \text{and} \quad \int_{-1}^{1} xdx,$$

$$\int_a^b 1dx = b - a = c_1 \qquad \int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 x_1$$

Since $c_1 = b - a,$ the other equation becomes

$$(b-a)x_1 = \frac{b^2 - a^2}{2} \qquad x_1 = \frac{b+a}{2}$$

Solution (cont.)

Therefore, one-point Gauss Quadrature Rule can be expressed as

$$\int_a^b f(x)dx \approx (b-a)f\left(\frac{b+a}{2}\right)$$

Example 2

A company advertises that every roll of their toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515 \quad e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515 \quad e^{-0.3881(y-252.2)^2} dy$$

a) Use two-point Gauss Quadrature Rule to find the probability.
b) Find the true error, E_t for part (a).
c) Also, find the absolute relative true error, $|\varepsilon_a|$ for part (a).

Solution

a) First, change the limits of integration from [250,270] to [-1,1] by previous relations as follows

$$\int_{250}^{270} f(y)dy = \frac{270-250}{2}\int_{-1}^{1} f\left(\frac{270-250}{2}y + \frac{270+250}{2}\right)dy$$

$$= 10\int_{-1}^{1} f(10y + 260)dy$$

Solution (cont)

Next, get weighting factors and function argument values from Table 1 for the two point rule,

$$c_1 = 1.000000000$$

$$y_1 = -0.577350269$$

$$c_2 = 1.000000000$$

$$y_2 = 0.577350269$$

Solution (cont.)

Now we can use the Gauss Quadrature formula

$$10\int_{-1}^{1} f\left(10y+260\right)dy \approx 10\left[c_1 f\left(10y_1+260\right)+c_2 f\left(10y_2+260\right)\right]$$

$$=10\left[f\left(10(-0.5773503)+260\right)+ f\left(10(0.5773503)+260\right)\right]$$

$$=10\left[f(254.23)+ f(265.77)\right]$$

$$=10\left[(0.071015)+(3.2237\times 10^{-32})\right]$$

$$=0.71015$$

Solution (cont)

since

$$f(254.23) = 0.3515e^{-0.3881(254.23-252.2)^2} = 0.071015$$

$$f(265.77) = 0.3515e^{-0.3881(265.77-252.2)^2} = 3.2237\times 10^{-32}$$

Solution (cont)

b) The true error, E_t , is

$$E_t = True\ Value - Approximate\ Value$$

$$= 0.97377 - 0.71015$$

$$= 0.26362$$

c) The absolute relative true error, $|\in_t|$, is (Exact value = 0.97377)

$$|\in_t| = \left|\frac{0.97377 - 0.71015}{0.97377}\right| \times 100\%$$

$$= 27.072\%$$