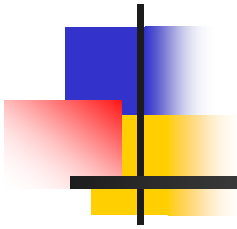


Integration



Topic: Romberg Rule

Major: Industrial Engineering

Basis of Romberg Rule

Integration

The process of measuring the area under a curve.

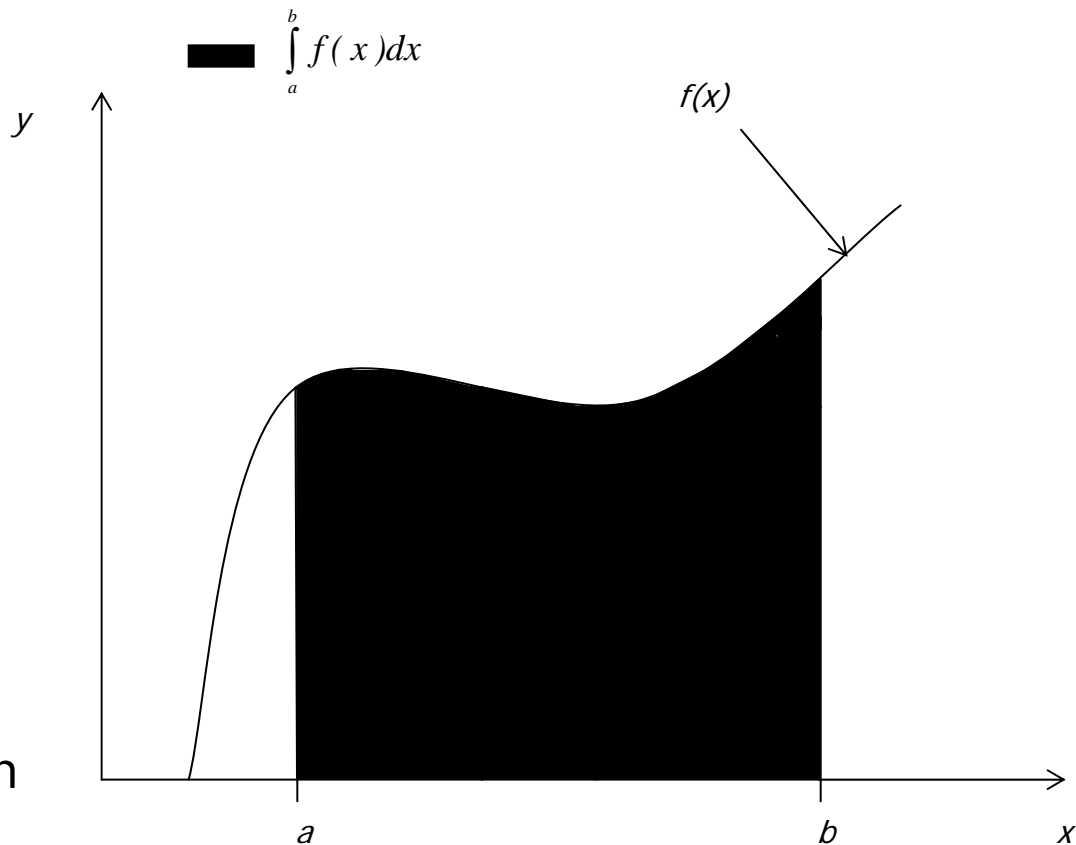
$$I = \int_a^b f(x) dx$$

Where:

$f(x)$ is the integrand

a = lower limit of integration

b = upper limit of integration





What is The Romberg Rule?

Romberg Integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integral by reducing the True Error.



Error in Multiple Segment Trapezoidal Rule

The true error in a multiple segment Trapezoidal Rule with n segments for an integral

$$I = \int_a^b f(x) dx$$

Is given by

$$E_t = \frac{(b-a)^3}{12n^2} \sum_{i=1}^n \frac{f''(\xi_i)}{n}$$

where for each i , ξ_i is a point somewhere in the domain, $[a + (i-1)h, a + ih]$.



Error in Multiple Segment Trapezoidal Rule

The term $\frac{\sum_{i=1}^n f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$.

This leads us to say that the true error, E_t previously defined can be approximated as

$$E_t \cong \alpha \frac{1}{n^2}$$



Error in Multiple Segment Trapezoidal Rule

Table 1 shows the results obtained for the integral using multiple segment Trapezoidal rule for

$$P(y \geq 250) = \int_{250}^{270} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

n	Value	E_t	$ \epsilon_t \%$	$ \epsilon_a \%$
1	0.53721	0.43666	44.832	---
2	0.26861	0.70516	72.416	99.999
3	0.18009	0.79368	81.506	49.153
4	0.21814	0.75562	77.598	17.447
5	0.50728	0.46648	47.905	56.997
6	0.80177	0.17200	17.663	36.729
7	0.93439	0.039381	4.0442	14.193
8	0.95767	0.016092	1.6525	2.4317

Table 1: Multiple Segment Trapezoidal Rule Values



Error in Multiple Segment Trapezoidal Rule

The true error gets approximately quartered as the number of segments is doubled. This information is used to get a better approximation of the integral, and is the basis of Richardson's extrapolation.



Richardson's Extrapolation for Trapezoidal Rule

The true error, E_t in the n -segment Trapezoidal rule is estimated as

$$E_t \cong \frac{C}{n^2}$$

where C is an *approximate constant* of proportionality. Since

$$E_t = TV - I_n$$

Where TV = true value and I_n = approx. value



Richardson's Extrapolation for Trapezoidal Rule

From the previous development, it can be shown that

$$\frac{C}{(2n)^2} \cong TV - I_{2n}$$

when the segment size is doubled and that

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

which is Richardson's Extrapolation.



Example 1

A company advertises that every roll of toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by :

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

- Use Richardson's rule to find the probability that there are 250 or more sheets. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.
- Find the true error, E_t for part (a).
- Find the absolute relative true error, $|\varepsilon_a|$ for part (a).



Solution

a) $I_2 = 0.26861$ $I_4 = 0.21814$

Using Richardson's extrapolation formula
for Trapezoidal rule

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3} \quad \text{and choosing } n=2,$$

$$\begin{aligned} TV &\cong I_4 + \frac{I_4 - I_2}{3} = 0.21814 + \frac{0.21814 - 0.26861}{3} \\ &= 0.20132 \end{aligned}$$



Solution (cont.)

- The exact value of the above integral cannot be found. We assume
- b) the value obtained by adaptive numerical integration using Maple as the exact value for calculating the true error and relative true error.

$$\begin{aligned} P(y \geq 250) &= \int_{250}^{270} 0.3515 e^{-0.3881(y-252.2)^2} dy \\ &= 0.97377 \end{aligned}$$

$$\begin{aligned} E_t &= \textit{True Value} - \textit{Approximate Value} \\ &= 0.97377 - 0.20132 \\ &= 0.77245 \end{aligned}$$



Solution (cont.)

c) The absolute relative true error $|\epsilon_t|$ would then be

$$|\epsilon_t| = \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 = \left| \frac{0.77245}{0.97377} \right| \times 100 = 79.326\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.



Solution (cont.)

Table 2: The values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$P(y \geq 250) = \int_{250}^{270} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

n	Trapezoidal Rule	ϵ_r for Trapezoidal Rule	Richardson's Extrapolation	ϵ_r for Richardson's Extrapolation
1	0.53721	44.832	--	--
2	0.26861	72.416	0.17908	81.609
4	0.21814	77.598	0.20132	79.326
8	0.95767	1.6525	1.2042	23.664

Table 2: Richardson's Extrapolation Values



Romberg Integration

Romberg integration is same as Richardson's extrapolation formula as given previously. However, Romberg used a recursive algorithm for the extrapolation. Recall

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

This can alternately be written as

$$(I_{2n})_R = I_{2n} + \frac{I_{2n} - I_n}{3} = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$



Romberg Integration

Note that the variable TV is replaced by $(I_{2n})_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign \cong is replaced by $=$ sign. Hence the estimate of the true value now is

$$TV \cong (I_{2n})_R + Ch^4$$

Where Ch^4 is an approximation of the true error.



Romberg Integration

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

It follows from the two previous expressions that the true value TV can be written as

$$\begin{aligned} TV &\cong (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15} \\ &= I_{4n} + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1} \end{aligned}$$



Romberg Integration

A general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index k represents the order of extrapolation. $k=1$ represents the values obtained from the regular Trapezoidal rule, $k=2$ represents values obtained using the true estimate as $O(h^2)$. The index j represents the more and less accurate estimate of the integral.



Example 2

A company advertises that every roll of toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

Use Romberg's rule to find the probability. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given.



Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = 0.53721$$

$$I_{1,2} = 0.26861$$

$$I_{1,3} = 0.21814$$

$$I_{1,4} = 0.95767$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively.



Solution (cont.)

To get the first order extrapolation values,

$$\begin{aligned}I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \\ &= 0.26861 + \frac{0.26861 - 0.53721}{3} \\ &= 0.17908\end{aligned}$$

Similarly,

$$\begin{aligned}I_{2,2} &= I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} \\ &= 0.21814 + \frac{0.21814 - 0.26861}{3} \\ &= 0.20132\end{aligned}$$

$$\begin{aligned}I_{2,3} &= I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3} \\ &= 0.95767 + \frac{0.95767 - 0.21814}{3} \\ &= 1.2042\end{aligned}$$



Solution (cont.)

For the second order extrapolation values,

$$\begin{aligned} I_{3,1} &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \\ &= 0.20132 + \frac{0.20132 - 0.17908}{15} \\ &= 0.20280 \end{aligned}$$

Similarly,

$$\begin{aligned} I_{3,2} &= I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15} \\ &= 1.2042 + \frac{1.2042 - 0.20132}{15} \\ &= 1.2711 \end{aligned}$$

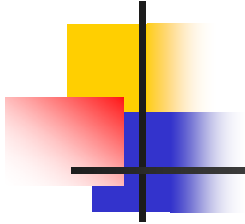


Solution (cont.)

For the third order extrapolation values,

$$\begin{aligned}I_{4,1} &= I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63} \\ &= 1.2711 + \frac{1.2711 - 0.20280}{63} \\ &= 1.2881\end{aligned}$$

Table 3 shows these increased correct values in a tree graph.



Solution (cont.)

Table 3: Improved estimates of the integral value using Romberg Integration

		1 st Order	2 nd Order	3 rd Order
1-segment	0.53721	0.17908	0.20280	1.2881
2-segment	0.26861			
4-segment	0.21814	1.2711		
8-segment	0.95767	1.2042		