Integration

Topic: Simpson's 1/3rd Rule

Major: Industrial Engineering

What is Integration?

Integration

The process of measuring the area under a curve.

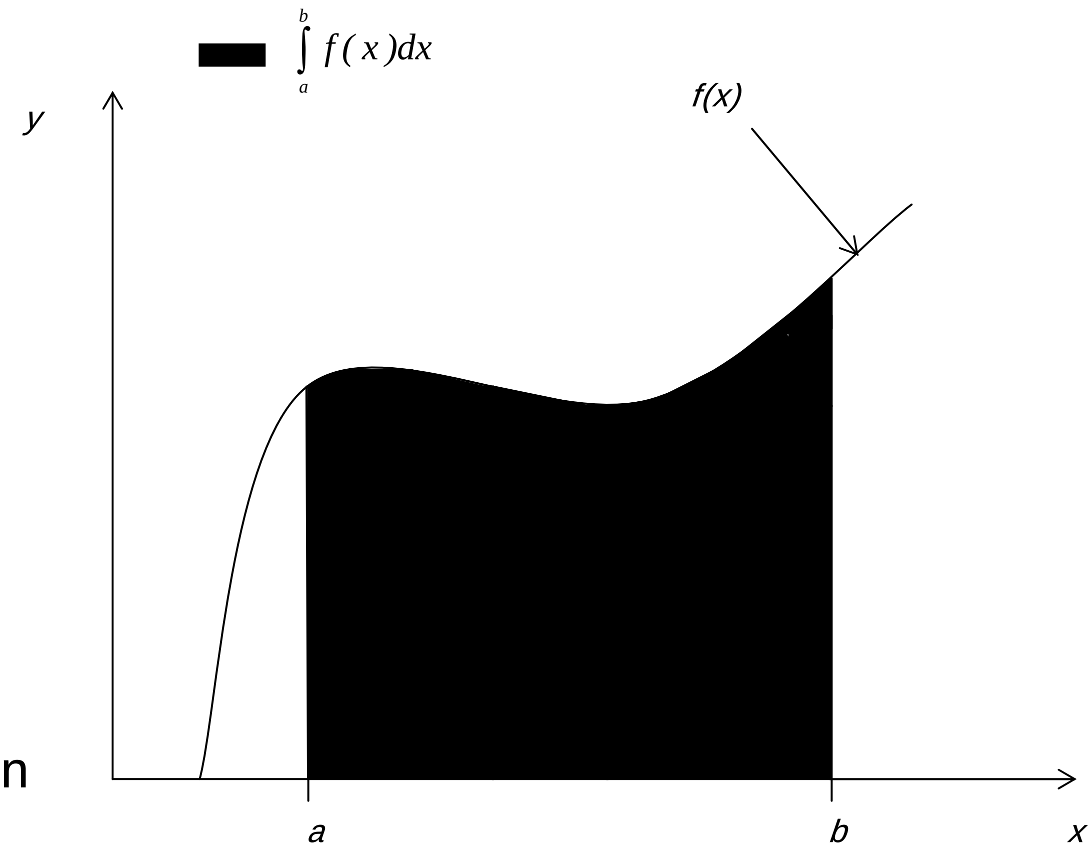
$$I = \int_a^b f(x)dx$$

Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration



Simpson's 1/3rd Rule

Basis of Simpson's 1/3rd Rule

Trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial in the interval of integration. Simpson's 1/3rd rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

Where $f_2(x)$ is a second order polynomial.

$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

Basis of Simpson's 1/3rd Rule

Choose

$$(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \text{ and } (b, f(b))$$

as the three points of the function to evaluate a_0, a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Basis of Simpson's 1/3rd Rule

Solving the previous equations for a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Basis of Simpson's 1/3rd Rule

Then

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Basis of Simpson's 1/3rd Rule

Substituting values of a_0, a_1, a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

Since for Simpson's 1/3rd Rule, the interval [a, b] is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Basis of Simpson's 1/3rd Rule

Hence

$$\int_a^b f_2(x)dx = \frac{h}{3}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

Because the above form has 1/3 in its formula, it is called Simpson's 1/3rd Rule.

Example 1

A company advertises that every roll of toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by :

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515\ e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515\ e^{-0.3881(y-252.2)^2} dy$$

a) Use Simpson's 1/3rd Rule to find the probability that there are 250 or more sheets.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\varepsilon_a|$ for part (a).

Solution

a) $$P(y \geq 250) = \int_{250}^{270} 0.3515\ e^{-0.3881(y-252.2)^2}\,dy$$

$$f(y) = 0.3515e^{-0.3881(y-252.2)^2}$$

$$P(y \geq 250) = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \left(\frac{270-250}{6}\right)\left[f(250) + 4f(260) + f(270)\right]$$

$$= \left(\frac{20}{6}\right)\left[0.053721 + 4(1.9559\times10^{-11}) + 1.3888\times10^{-54}\right]$$

$$= 0.17907$$

Solution (cont)

b) The exact value of the above integral cannot be found. We assume the value obtained by adaptive numerical integration using Maple as the exact value for calculating the true error and relative true error.

$$P(y \geq 250) = \int_{250}^{270} 0.3515 \; e^{-0.3881(y-252.2)^2} dy$$

$$= 0.97377$$

True Error

$$E_t = True\ Value - Approximate\ Value$$

$$= 0.97377 - 0.17907$$

$$= 0.79470$$

Solution (cont)

c) Absolute relative true error,

$$\left|\in_t\right| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{0.79479}{0.97377}\right| \times 100\%$$

$$= 81.611\%$$

Multiple Segment Simpson's 1/3rd Rule

Multiple Segment Simpson's 1/3rd Rule

Just like in multiple segment Trapezoidal Rule, one can subdivide the interval [a, b] into n segments and apply Simpson's 1/3rd Rule repeatedly over every two segments. Note that n needs to be even. Divide interval [a, b] into equal segments, hence the segment width

$$h=\frac{b-a}{n} \qquad \int_{a}^{b} f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

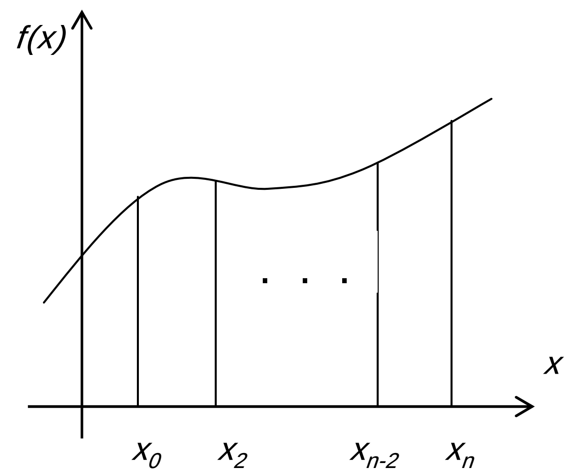
where

$$x_0 = a \qquad x_n = b$$

Multiple Segment Simpson's 1/3rd Rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + ...$$

$$.... + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$



Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx = (x_2 - x_0)\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + ...$$

$$+ (x_4 - x_2)\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + ...$$

Multiple Segment Simpson's 1/3rd Rule

$$\ldots + (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6}\right] + \ldots$$

$$+ (x_n - x_{n-2})\left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right]$$

Since

$$x_i - x_{i-2} = 2h \qquad i = 2, 4, \ldots, n$$

Multiple Segment Simpson's 1/3rd Rule

Then

$$\int_a^b f(x)dx = 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Multiple Segment Simpson's 1/3rd Rule

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4\{f(x_1) + f(x_3) + \ldots + f(x_{n-1})\} + \ldots\right]$$

$$\ldots + 2\{f(x_2) + f(x_4) + \ldots + f(x_{n-2})\} + f(x_n)\}]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n)\right]$$

$$= \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n)\right]$$

Example 2

A company advertises that every roll of toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by :

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515\ e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515\ e^{-0.3881(y-252.2)^2} dy$$

a) Use 4-segment Simpson's 1/3rd Rule to find the probability that there are 250 or more sheets.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\varepsilon_a|$ for part (a).

Solution

a) Using n segment Simpson's 1/3rd Rule,

$$h = \frac{b-a}{n} = \frac{270-250}{4} = 5$$

So

$$f(y_0) = f(250)$$

$$f(y_1) = f(250+5) = f(255)$$

$$f(y_2) = f(255+5) = f(260)$$

$$f(y_3) = f(260+5) = f(265)$$

$$f(y_4) = f(270)$$

Solution (cont.)

$$P(y \geq 250) = \frac{b-a}{3n}\left[f(y_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(y_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(y_i) + f(y_n) \right]$$

$$= \frac{270-250}{3(4)}\left[f(250) + 4\sum_{\substack{i=1 \\ i=odd}}^{3} f(y_i) + 2\sum_{\substack{i=2 \\ i=even}}^{2} f(y_i) + f(270) \right]$$

$$= \frac{20}{12}\left[f(250) + 4f(y_1) + 4f(y_3) + 2f(y_2) + f(270) \right]$$

Solution (cont.)

cont.

$$=\frac{10}{6}\left[f(250)+4f(255)+4f(265)+2f(260)+f(270)\right]$$

$$=\frac{10}{6}\left[0.053720+4(0.016769)+4(8.5259\times10^{-29})+2(1.9559\times10^{-11})+1.3888\times10^{-54}\right]$$

$$=0.20133$$

Solution (cont.)

b) In this case, the true error is

$$E_t = True\ Value - Approximate\ Value = 0.97377 - 0.20133 = 0.77244$$

c) The absolute relative true error

$$\left|\in_t\right| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{0.77244}{0.97377}\right| \times 100\%$$

$$= 79.325\%$$

Solution (cont.)

Table 1: Values of Simpson's 1/3rd Rule for Example 2 with multiple segments

n	Approximate Value	E_t	$\|\in_t\|$
2	0.17907	0.79470	81.611%
4	0.20133	0.77244	79.325%
6	1.0089	-0.035226	3.6175%
8	1.2042	-0.23042	23.663%
10	1.0954	-0.12167	12.495%

Error in the Multiple Segment Simpson's 1/3rd Rule

The true error in a single application of Simpson's 1/3rd Rule is given as

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In Multiple Segment Simpson's 1/3rd Rule, the error is the sum of the errors in each application of Simpson's 1/3rd Rule. The error in n segment Simpson's 1/3rd Rule is given by

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1) = -\frac{h^5}{90} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2) = -\frac{h^5}{90} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$

Error in the Multiple Segment Simpson's 1/3rd Rule

$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i) = -\frac{h^5}{90} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$

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$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right) = -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{4}\left(\zeta_{\frac{n}{2}}\right) = -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), \quad x_{n-2} < \zeta_{\frac{n}{2}} < x_n$$

Error in the Multiple Segment Simpson's 1/3rd Rule

Hence, the total error in Multiple Segment Simpson's 1/3rd Rule is

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i \quad = -\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \quad = -\frac{(b-a)^5}{90n^5}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^4}\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

Error in the Multiple Segment Simpson's 1/3rd Rule

The term $\dfrac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$ is an approximate average value of

$$f^{(4)}(x),\ a < x < b$$

Hence

$$E_t = -\frac{(b-a)^5}{90n^4}\overline{f}^{(4)}$$

where

$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$