

## Chapter 07.04

# Romberg Rule of Integration

*After reading this chapter, you should be able to:*

1. *derive the Romberg rule of integration, and*
2. *use the Romberg rule of integration to solve problems.*

### **What is integration?**

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.

Here, we will discuss the Romberg rule of approximating integrals of the form

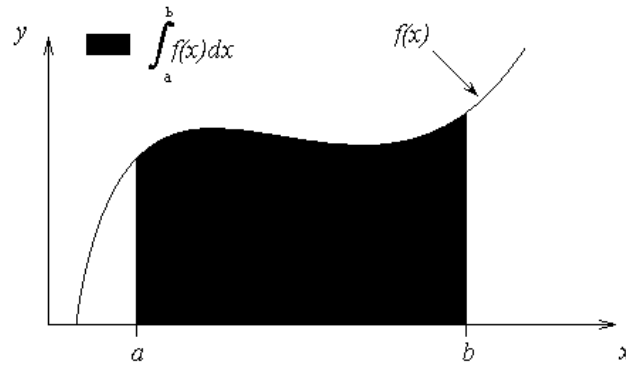
$$I = \int_a^b f(x)dx \quad (1)$$

where

$f(x)$  is called the integrand

$a$  = lower limit of integration

$b$  = upper limit of integration



**Figure 1** Integration of a function.

### Error in Multiple-Segment Trapezoidal Rule

The true error obtained when using the multiple segment trapezoidal rule with  $n$  segments to approximate an integral

$$\int_a^b f(x) dx$$

is given by

$$E_t = -\frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^n f''(\xi_i)}{n} \quad (2)$$

where for each  $i$ ,  $\xi_i$  is a point somewhere in the domain  $[a + (i-1)h, a + ih]$ , and

the term  $\frac{\sum_{i=1}^n f''(\xi_i)}{n}$  can be viewed as an approximate average value of  $f''(x)$  in  $[a, b]$ . This leads us to say that the true error  $E_t$  in Equation (2) is approximately proportional to

$$E_t \approx \alpha \frac{1}{n^2} \quad (3)$$

for the estimate of  $\int_a^b f(x) dx$  using the  $n$ -segment trapezoidal rule.

Table 1 shows the results obtained for

$$\int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

using the multiple-segment trapezoidal rule.

**Table 1** Values obtained using multiple segment trapezoidal rule for

$$x = \int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt .$$

$n$	Approximate Value	$E_t$	$ \epsilon_t \%$	$ \epsilon_a \%$
1	11868	-807	7.296	---
2	11266	-205	1.854	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

The true error for the 1-segment trapezoidal rule is  $-807$ , while for the 2-segment rule, the true error is  $-205$ . The true error of  $-205$  is approximately a quarter of  $-807$ . The true error gets approximately quartered as the number of segments is doubled from 1 to 2. The same trend is observed when the number of segments is doubled from 2 to 4 (the true error for 2-segments is  $-205$  and for four segments is  $-51.5$ ). This follows Equation (3). This information, although interesting, can also be used to get a better approximation of the integral. That is the basis of Richardson's extrapolation formula for integration by the trapezoidal rule.

### Richardson's Extrapolation Formula for Trapezoidal Rule

The true error,  $E_t$ , in the  $n$ -segment trapezoidal rule is estimated as

$$E_t \approx \alpha \frac{1}{n^2}$$

$$E_t \approx \frac{C}{n^2} \tag{4}$$

where  $C$  is an approximate constant of proportionality.

Since

$$E_t = TV - I_n \tag{5}$$

where

$TV$  = true value

$I_n$  = approximate value using  $n$ -segments

Then from Equations (4) and (5),

$$\frac{C}{n^2} \approx TV - I_n \tag{6}$$

If the number of segments is doubled from  $n$  to  $2n$  in the trapezoidal rule,

$$\frac{C}{(2n)^2} \approx TV - I_{2n} \tag{7}$$

Equations (6) and (7) can be solved simultaneously to get

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3} \quad (8)$$

### Example 1

A company advertises that every roll of toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

**Table 2** Values obtained using Trapezoidal rule.

$n$	Trapezoidal Rule
1	0.53721
2	0.26861
4	0.21814
8	0.95767

- Use Romberg's rule to find the probability. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 2.
- Find the true error,  $E_t$ , for part (a).
- Find the absolute relative true error,  $|\epsilon_t|$ , for part (a).

### Solution

$$\begin{aligned} \text{a)} \quad I_2 &= 0.26861 \\ I_4 &= 0.21814 \end{aligned}$$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing  $n = 2$ ,

$$\begin{aligned} TV &\approx I_4 + \frac{I_4 - I_2}{3} \\ &\approx 0.21814 + \frac{0.21814 - 0.26861}{3} \\ &\approx 0.20132 \end{aligned}$$

b) The exact value of the above integral cannot be found. For calculating the true error and relative true error, we assume the value obtained by adaptive numerical integration using Maple as the exact value.

$$\begin{aligned} P(y \geq 250) &= \int_{250}^{270} 0.3515 e^{-0.3881(y-252.2)^2} dy \\ &= 0.97377 \end{aligned}$$

so the true error is,

$$\begin{aligned}
 E_t &= \text{True Value} - \text{Approximate Value} \\
 &= 0.97377 - 0.20132 \\
 &= 0.77245
 \end{aligned}$$

c) The absolute relative true error,  $|\epsilon_t|$ , would then be

$$\begin{aligned}
 |\epsilon_t| &= \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \% \\
 &= \left| \frac{0.77245}{0.97377} \right| \times 100 \% \\
 &= 79.326 \%
 \end{aligned}$$

Table 3 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

**Table 3** Values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$P(y \geq 250) = \int_{250}^{270} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

$n$	Trapezoidal Rule	$ \epsilon_t $ for Trapezoidal Rule %	Richardson's Extrapolation	$ \epsilon_t $ for Richardson's Extrapolation %
1	0.53721	44.832	--	--
2	0.26861	72.416	0.17908	81.610
4	0.21814	77.598	0.20132	79.326
8	0.95767	1.6525	1.2042	23.662

### Romberg Integration

Romberg integration is the same as Richardson's extrapolation formula as given by Equation (8). However, Romberg used a recursive algorithm for the extrapolation as follows.

The estimate of the true error in the trapezoidal rule is given by

$$E_t = -\frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^n f''(\xi_i)}{n}$$

Since the segment width,  $h$ , is given by

$$h = \frac{b-a}{n}$$

Equation (2) can be written as

$$E_t = -\frac{h^2(b-a)}{12} \frac{\sum_{i=1}^n f''(\xi_i)}{n} \quad (9)$$

The estimate of true error is given by

$$E_t \approx Ch^2 \quad (10)$$

It can be shown that the exact true error could be written as

$$E_t = A_1 h^2 + A_2 h^4 + A_3 h^6 + \dots \quad (11)$$

and for small  $h$ ,

$$E_t = A_1 h^2 + O(h^4) \quad (12)$$

Since we used  $E_t \approx Ch^2$  in the formula (Equation (12)), the result obtained from Equation (10) has an error of  $O(h^4)$  and can be written as

$$\begin{aligned} (I_{2n})_R &= I_{2n} + \frac{I_{2n} - I_n}{3} \\ &= I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1} \end{aligned} \quad (13)$$

where the variable  $TV$  is replaced by  $(I_{2n})_R$  as the value obtained using Richardson's extrapolation formula. Note also that the sign  $\approx$  is replaced by the sign  $=$ .

Hence the estimate of the true value now is

$$TV \approx (I_{2n})_R + Ch^4$$

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3} \quad (14)$$

then

$$TV \approx (I_{4n})_R + C\left(\frac{h}{2}\right)^4$$

From Equation (13) and (14),

$$\begin{aligned} TV &\approx (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15} \\ &= (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1} \end{aligned} \quad (15)$$

The above equation now has the error of  $O(h^6)$ . The above procedure can be further improved by using the new values of the estimate of the true value that has the error of  $O(h^6)$  to give an estimate of  $O(h^8)$ .

Based on this procedure, a general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, \quad k \geq 2 \quad (16)$$

The index  $k$  represents the order of extrapolation. For example,  $k=1$  represents the values obtained from the regular trapezoidal rule,  $k=2$  represents the values obtained using the true error estimate as  $O(h^2)$ , etc. The index  $j$  represents the more and less accurate estimate of the integral. The value of an integral with a  $j+1$  index is more accurate than the value of the integral with a  $j$  index.

For  $k = 2, j = 1,$

$$\begin{aligned} I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{4^{2-1} - 1} \\ &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \end{aligned}$$

For  $k = 3, j = 1,$

$$\begin{aligned} I_{3,1} &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{4^{3-1} - 1} \\ &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \end{aligned} \tag{17}$$

### Example 2

A company advertises that every roll of toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

Use Romberg's rule to find the probability. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given.

### Solution

From Table 2, the needed values from original Trapezoidal rule are

$$I_{1,1} = 0.53721$$

$$I_{1,2} = 0.26861$$

$$I_{1,3} = 0.21814$$

$$I_{1,4} = 0.95767$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively. To get the first order extrapolation values,

$$\begin{aligned} I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \\ &= 0.26861 + \frac{0.26861 - 0.53721}{3} \\ &= 0.17908 \end{aligned}$$

Similarly

$$\begin{aligned} I_{2,2} &= I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} \\ &= 0.21814 + \frac{0.21814 - 0.26861}{3} \\ &= 0.20132 \end{aligned}$$

$$\begin{aligned}
 I_{2,3} &= I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3} \\
 &= 0.95767 + \frac{0.95767 - 0.21814}{3} \\
 &= 1.2042
 \end{aligned}$$

For the second order extrapolation values,

$$\begin{aligned}
 I_{3,1} &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \\
 &= 0.20132 + \frac{0.20132 - 0.17908}{15} \\
 &= 0.20280
 \end{aligned}$$

Similarly

$$\begin{aligned}
 I_{3,2} &= I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15} \\
 &= 1.2042 + \frac{1.2042 - 0.20132}{15} \\
 &= 1.2710
 \end{aligned}$$

For the third order extrapolation values,

$$\begin{aligned}
 I_{4,1} &= I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63} \\
 &= 1.2710 + \frac{1.2710 - 0.20280}{63} \\
 &= 1.2880
 \end{aligned}$$

Table 4 shows these increased correct values in a tree graph.

**Table 4** Improved estimates of value of integral using Romberg integration.

		1 <sup>st</sup> Order	2 <sup>nd</sup> Order	3 <sup>rd</sup> Order
1-segment	0.53721	0.17908	0.20280	1.2880
2-segment	0.26861			
4-segment	0.21814	0.20132		
8-segment	0.95767	1.2042		

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**INTEGRATION**

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Topic	Romberg Rule
Summary	Textbook notes of Romberg Rule of integration.
Major	Industrial Engineering
Authors	Autar Kaw
Date	August 27, 2009
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>

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