Differentiation-Discrete Functions

Major: Mechanical Engineering

Authors: Autar Kaw, Sri Harsha Garapati

Forward Difference Approximation

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For a finite 'Δx'

$$f'(x) \cong \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Graphical Representation Of Forward Difference Approximation

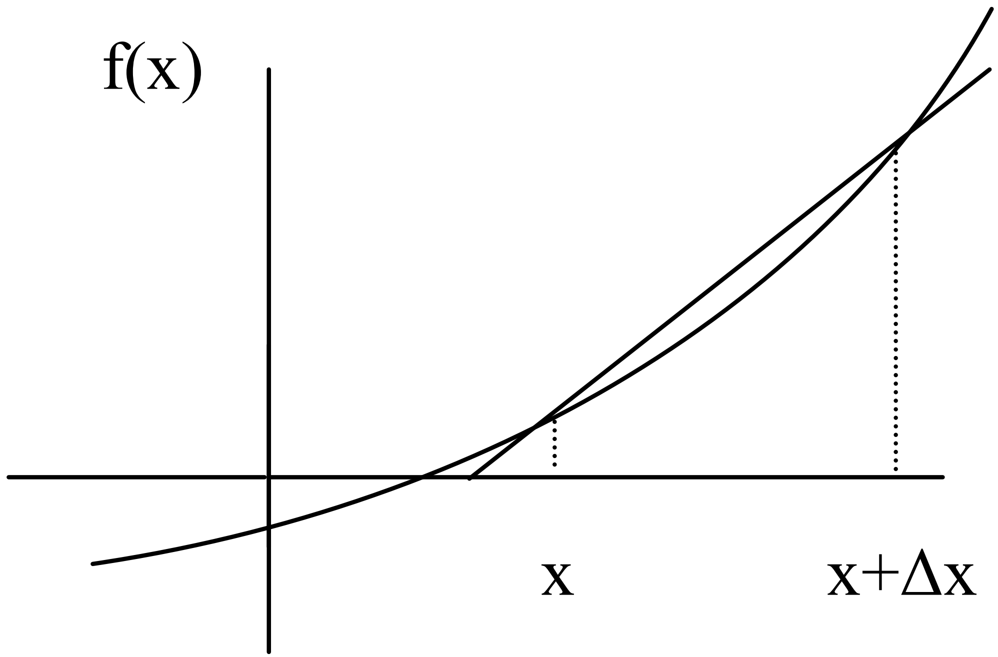

Figure 1: Graphical Representation of forward difference approximation of first derivative

-340^0 F

Example 1

To find contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 1.

a) Is the rate of change of coefficient of thermal expansion with respect to temperature more at T=80ºF than at T=-340ºF?

b) The data given in the Table 1 can be regressed $\alpha = a_0 + a_1 T + a_2 T^2$ to get $\alpha = 6.0217 \times 10^{-6} + 6.2782 \times 10^{-9} \mathrm{T} - 1.2218 \times 10^{-11} \mathrm{T}^2$ Compare the results with part (a) if you used the regression curve to find the rate of change of coefficient of thermal expansion with respect to temperature at T=80ºF than at T=340ºF

Example 1 Cont.

Temperature, T (^{o}F)	**Coefficient of thermal expansion, α** (in/in/^{o}F)
80	$6.47\ 10^{-6}$
40	$6.24\ 10^{-6}$
-40	$5.72\ 10^{-6}$
-120	$5.09\ 10^{-6}$
-200	$4.30\ 10^{-6}$
-280	$3.33\ 10^{-6}$
-340	$2.45\ 10^{-6}$

Table 1: Coefficient of Thermal Expansion as a function of Temperature

Example 1 Cont.

a) Using Backward Divided difference approximation method at $T = 80^0 F$

$$\frac{d\alpha(T_i)}{dT} \cong \frac{\alpha(T_i) - \alpha(T_{i-1})}{\Delta T}$$

$$T_i = 80$$

$$T_{i-1} = T_i - \Delta T$$

$$T_{i+1} = 80 - 40 = 40$$

$$\frac{d\alpha(80)}{dT} \approx \frac{\alpha(80) - \alpha(40)}{40}$$

$$= \frac{6.47 \times 10^{-6} - 6.24 \times 10^{-6}}{40}$$

$$= 5.75 \times 10^{-9} \, in/in/{}^oF^2$$

Example 1 Cont.

Using Forward Divided difference approximation method at $T = -340^0 F$

$$\frac{d\alpha(T_i)}{dT} \cong \frac{\alpha(T_{i+1}) - \alpha(T_i)}{\Delta T}$$

$$T_i = -340$$

$$t_{i+1} = t_i + \Delta t$$

$$t_{i+1} = -340 + 60 = -280$$

$$\frac{d\alpha(80)}{dT} \approx \frac{\alpha(-280) - \alpha(-340)}{60}$$

$$= \frac{3.33 \times 10^{-6} - 2.45 \times 10^{-6}}{60}$$

$$= 0.147 \times 10^{-9} in/in/^oF^2$$

From the above two results it is clear that the rate of change of coefficient of thermal expansion is more at $T = 80^0 F$ than $T = -340^0 F$

Example 1 Cont.

b)**Given**

$$\alpha = 6.0217\times10^{-6} + 6.2782\times10^{-9}\,T - 1.2218\times10^{-11}\,T^{2}$$

$$\frac{d\alpha}{dT} = 6.2782\times10^{-9} - 2.4436\times10^{-11}\,T$$

$$\frac{d\alpha(80)}{dT} = 6.2782\times10^{-9} - 2.4436\times10^{-11}(80)$$

$$= 4.3233\times10^{-9}\,in/in/^{o}F^{2}$$

$$\frac{d\alpha(-340)}{dT} = 6.2782\times10^{-9} - 2.4436\times10^{-11}(-340)$$

$$= 0.14586\times10^{-9}\,in/in/^{o}F^{2}$$

Change in Coefficient of Thermal Expansion	Divided Difference Approximation	Regression
$\frac{d\alpha}{dT}$, $T = 80^{0}F$	$5.75\times10^{-9}\,in/in/^{o}F^{2}$	$4.3233\times10^{-9}\,in/in/^{o}F^{2}$
$\frac{d\alpha}{dT}$, $T = -340^{0}F$	$0.147\times10^{-9}\,in/in/^{o}F^{2}$	$0.14586\times10^{-9}\,in/in/^{o}F^{2}$

Direct Fit Polynomials

In this method, given $'n+1'$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1 x + \ldots\ldots + a_{n-1}x^{n-1} + a_n x^n$$

To find the first derivative,

$$P_n'(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \ldots\ldots + (n-1)a_{n-1}x^{n-2} + na_n x^{n-1}$$

Similarly other derivatives can be found.

Example 2-Direct Fit Polynomials

To find contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 2.

a)Using the third order polynomial interpolant, find the change in coefficient of thermal expansion at T=80ºF and T=-340ºF. Compare these results with Example 1.

b) The data given in the Table 1 can be regressed to $\alpha = a_0 + a_1T + a_2T^2$ to get $\alpha = 6.0217 \times 10^{-6} + 6.2782 \times 10^{-9}\mathrm{T} - 1.2218 \times 10^{-11}\mathrm{T}^2$ Compare the results with part (a) if you used the regression curve to find the rate of change of coefficient of thermal expansion with respect to temperature at T=80ºF than at T=-340ºF

Example 2-Direct Fit Polynomials cont.

Temperature, T (°F)	**Coefficient of thermal expansion, α** (in/in/°F)
80	$6.47\ 10^{-6}$
40	$6.24\ 10^{-6}$
-40	$5.72\ 10^{-6}$
-120	$5.09\ 10^{-6}$
-200	$4.30\ 10^{-6}$
-280	$3.33\ 10^{-6}$
-340	$2.45\ 10^{-6}$

Table 2: Coefficient of Thermal Expansion as a function of Temperature

Example 2-Direct Fit Polynomials cont.

For the third order polynomial (also called cubic interpolation), we choose the velocity given by

$$\alpha(t) = a_0 + a_1 T + a_2 T^2 + a_3 T^3$$

Change in Thermal Expansion Coefficient at 80^{0}F

Since we want to find the voltage at T=80^{0}F, and we are using a third order polynomial, we need to choose the four points closest to $T = 80^0 F$ and that also bracket $T = 80^0 F$ to evaluate it.

The four points are $T_0 = 80^0 F, T_1 = 40^0 F, T_2 = -40^0 F \; \& \; T_3 = -120^0 F$

$$T_o = 80^0 F, \;\; \alpha(T_o) = 6.47 \times 10^{-6}$$
$$T_1 = 40^0 F, \;\; \alpha(T_1) = 6.24 \times 10^{-6}$$
$$T_2 = -40^0 F, \;\; \alpha(T_2) = 5.72 \times 10^{-6}$$
$$T_3 = -120^0 F, \;\; \alpha(T_3) = 5.09 \times 10^{-6}$$

Example 2-Direct Fit Polynomials cont.

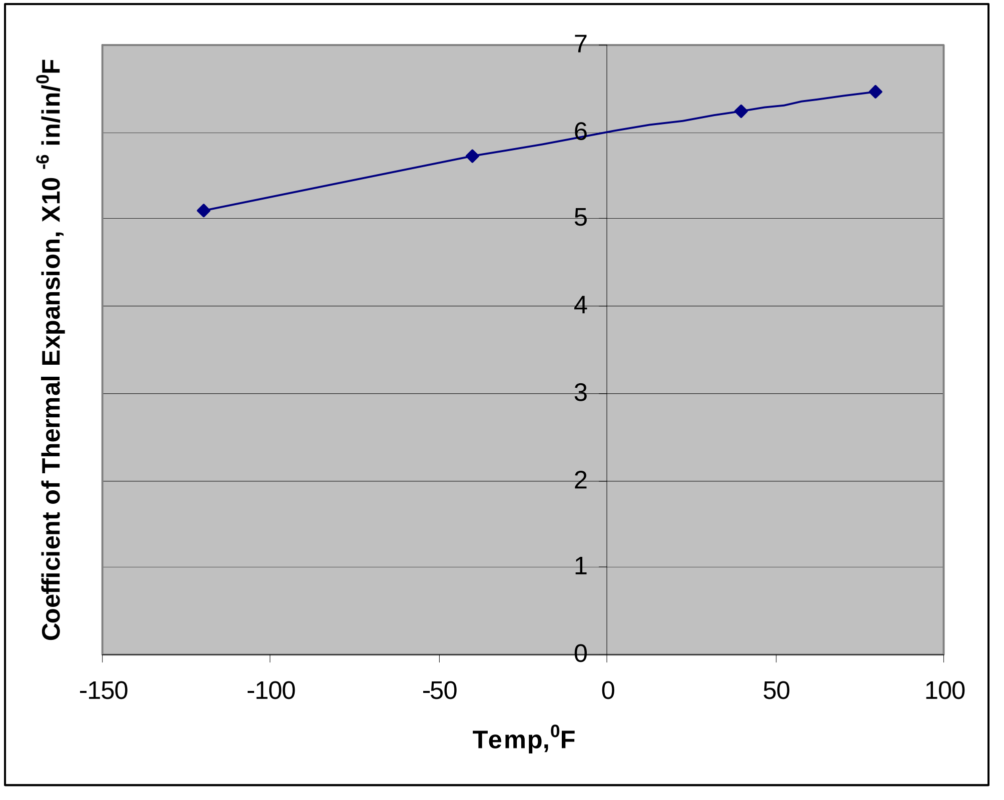

Figure 2: Graph of Coefficient of Thermal expansion vs. Temperature

Example 2-Direct Fit Polynomials cont.

such that

$$\alpha(80) = 6.47 \times 10^{-6} = a_0 + a_1(80) + a_2(80)^2 + a_3(80)^3$$

$$\alpha(40) = 6.24 \times 10^{-6} = a_0 + a_1(40) + a_2(40)^2 + a_3(40)^3$$

$$\alpha(-40) = 5.72 \times 10^{-6} = a_0 + a_1(-40) + a_2(-40)^2 + a_3(-40)^3$$

$$\alpha(-120) = 5.09 \times 10^{-6} = a_0 + a_1(-120) + a_2(-120)^2 + a_3(-120)^3$$

Writing the four equations in matrix form, we have

Example 2-Direct Fit Polynomials cont.

$$\begin{bmatrix} 1 & 80 & 6400 & 512000 \\ 1 & 40 & 1600 & 64000 \\ 1 & -40 & 1600 & -64000 \\ 1 & -120 & 14400 & -1728000 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 6.47 \times 10^{-6} \\ 6.24 \times 10^{-6} \\ 5.72 \times 10^{-6} \\ 5.09 \times 10^{-6} \end{bmatrix}$$

Solving the above four equations gives

$$a_0 = 0.59915 \times 10^{-5}$$

$$a_1 = 0.648125 \times 10^{-8}$$

$$a_2 = -0.71875 \times 10^{-11}$$

$$a_3 = 0.11719 \times 10^{-13}$$

Example 2-Direct Fit Polynomials cont.

Hence

$$\alpha(T) = a_0 + a_1 T + a_2 T^2 + a_3 T^3$$

$$= 0.59915\times10^{-5} + 0.64812\times10^{-8}T - 0.71875\times10^{-11}T^2 + 0.11719\times10^{-13}T^3, \quad -120 \le T \le 80$$

The change in coefficient of thermal expansion at T=80^{0}F is given by

$$\frac{d\alpha(80)}{dT} = \frac{d}{dt}\alpha(T)\Big|_{T=80}$$

,

Given that

$$\alpha(T) = 0.59915\times10^{-5} + 0.64812\times10^{-8}T - 0.71875\times10^{-11}T^2 + 0.11719\times10^{-13}T^3, \quad -120 \le T \le 80$$

$$\frac{d\alpha(T)}{dT} = \frac{d}{dT}\alpha(T)$$

$$= \frac{d}{dt}\left(0.59915\times10^{-5} + 0.64812\times10^{-8}T - 0.71875\times10^{-11}T^2 + 0.11719\times10^{-13}T^3\right)$$

Example 2-Direct Fit Polynomials cont.

$$= 0.64812\times10^{-8} - 1.4375\times10^{-11}T + 0.35157\times10^{-13}T^2, \quad -120 \le T \le 80$$

$$\frac{d\alpha(80)}{dT} = 0.64812\times10^{-8} - 1.4375\times10^{-11}(80) + 0.35157\times10^{-13}(80)^2$$

$$= 5.5562\times10^{-9}\, in/in/{}^{o}F^2$$

Example 2-Direct Fit Polynomials cont.

Change in Themal Expansion Coefficient at -340^{0}F

Since we want to find the rate of change in thermal expansion at $T=-340^0F$, and we are using a third order polynomial, we need to choose the four points closest to $T=-340^0F$ and that also bracket $T=-340^0F$ to evaluate it.

The four points are

$$T_0=-120^0F, T_1=-200^0F, T_2=-280^0F \; \& \; T_3=-340^0F$$

$$T_o=-120^0F,\ \alpha(T_o)=5.09\times10^{-6}$$

$$T_1=-200^0F,\ \alpha(T_1)=4.30\times10^{-6}$$

$$T_2=-280^0F,\ \alpha(T_2)=3.33\times10^{-6}$$

$$T_3=-340^0F,\ \alpha(T_3)=2.45\times10^{-6}$$

Example 2-Direct Fit Polynomials cont.

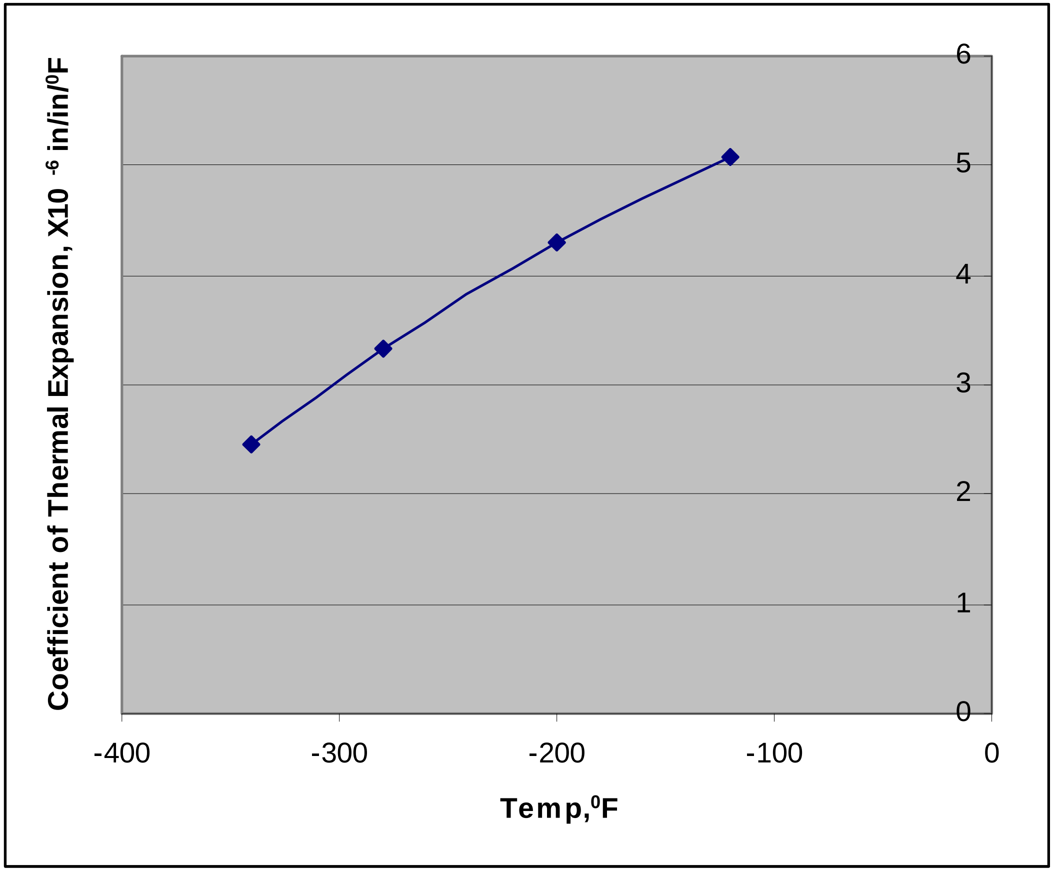

Figure 3: Graph of Coefficient of Thermal Expansion vs. Temperature

Example 2-Direct Fit Polynomials cont.

Such that

$$\alpha(-120) = 5.09\times10^{-6} = a_0 + a_1(-120) + a_2(-120)^2 + a_3(-120)^3$$

$$\alpha(-200) = 4.30\times10^{-6} = a_0 + a_1(-200) + a_2(-200)^2 + a_3(-200)^3$$

$$\alpha(-280) = 3.33\times10^{-6} = a_0 + a_1(-280) + a_2(-280)^2 + a_3(-280)^3$$

$$\alpha(-340) = 2.45\times10^{-6} = a_0 + a_1(-340) + a_2(-340)^2 + a_3(-340)^3$$

Writing the four equations in matrix form, we have

Example 2-Direct Fit Polynomials cont.

$$\begin{bmatrix} 1 & -120 & 14400 & -1728000 \\ 1 & -200 & 40000 & -8000000 \\ 1 & -280 & 78400 & -21952000 \\ 1 & -340 & 115600 & -39304000 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5.09 \times 10^{-6} \\ 4.30 \times 10^{-6} \\ 3.33 \times 10^{-6} \\ 2.45 \times 10^{-6} \end{bmatrix}$$

Solving the above four equations gives

$$a_0 = 0.60625 \times 10^{-5}$$

$$a_1 = 0.74810 \times 10^{-8}$$

$$a_2 = -0.29018 \times 10^{-11}$$

$$a_3 = 0.18601 \times 10^{-13}$$

Example 2-Direct Fit Polynomials cont.

Hence

$$\alpha(T) = a_0 + a_1 T + a_2 T^2 + a_3 T^3$$

$$= 0.60625\times10^{-5} + 0.74810\times10^{-8}T - 0.29018\times10^{-11}T^2 + 0.18601\times10^{-13}T^3, \quad -340 \le T \le -120$$

The change in coefficient of thermal expansion at T=-340^{0}F is given by

$$\frac{d\alpha(-340)}{dT} = \frac{d}{dt}\alpha(T)\Big|_{T=-340}$$

Given that

$$\alpha(T) = 0.60625\times10^{-5} + 0.74810\times10^{-8}T - 0.29018\times10^{-11}T^2 + 0.18601\times10^{-13}T^3, \quad -340 \le T \le -120$$

Example 2-Direct Fit Polynomials cont.

$$\frac{d\alpha(T)}{dT} = \frac{d}{dT}\alpha(T)$$

$$= \frac{d}{dt}\left(0.60625\times10^{-5} + 0.74810\times10^{-8}T - 0.29018\times10^{-11}T^2 + 0.18601\times10^{-13}T^3\right)$$

$$= 0.74810\times10^{-8} - 0.58036\times10^{-11}T + 0.55803\times10^{-13}T^2, \quad -340 \le t \le 80$$

$$\frac{d\alpha(-340)}{dT} = 0.74810\times10^{-8} - 0.58036\times10^{-11}(-340) + 0.55803\times10^{-13}(-340)^2$$

$$= 0.15905\times10^{-9}\, in/in/{}^oF^2$$

Change in Coefficient of Thermal Expansion	Divided Difference Approximation	Regression
$\frac{d\alpha}{dT}$, $T = 80^0F$	$5.5562\times10^{-9}\, in/in/{}^oF^2$	$4.3233\times10^{-9}\, in/in/{}^oF^2$
$\frac{d\alpha}{dT}$, $T = -340^0F$	$0.15905\times10^{-9}\, in/in/{}^oF^2$	$0.14586\times10^{-9}\, in/in/{}^oF^2$

Lagrange Polynomial

In this method, given $(x_1, y_1), \ldots, (x_n, y_n)$, one can fit a $(n-1)^{th}$ order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Lagrange Polynomial Cont.

Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ is

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating equation (2) gives

Lagrange Polynomial Cont.

$$f_2'(x)=\frac{2x-(x_1+x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{2x-(x_0+x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{2x-(x_0+x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x)=\frac{2}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{2}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{2}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

Example 3

To find contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 3.

a)Using the third order polynomial interpolant, find the change in coefficient of thermal expansion at T=80°F and T=-340°F. Compare these results with Example 1.

b) The data given in the Table 1 can be regressed to $\alpha = a_0 + a_1 T + a_2 T^2$ to get

if you used the regression curve to find the rate of change of coefficient of thermal expansion with respect to temperature at T=80°F than at T=-340°F

Example 3 Cont.

Temperature, T (^oF)	**Coefficient of thermal expansion, α** (in/in/^oF)
80	$6.47\ 10^{-6}$
40	$6.24\ 10^{-6}$
-40	$5.72\ 10^{-6}$
-120	$5.09\ 10^{-6}$
-200	$4.30\ 10^{-6}$
-280	$3.33\ 10^{-6}$
-340	$2.45\ 10^{-6}$

Table 3: Coefficient of Thermal Expansion as a function of Temperature

Example 3 Cont.

Solution:

a) $$\alpha(T)=\left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)\alpha(T_0)+\left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)\alpha(T_1)+\left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)\alpha(T_2)$$

$$\frac{d\alpha}{dT}(T)=\frac{2T-(T_1+T_2)}{(T_0-T_1)(T_0-T_2)}\alpha(T_0)+\frac{2T-(T_0+T_2)}{(T_1-T_0)(T_1-T_2)}\alpha(T_1)+\frac{2T-(T_0+T_1)}{(T_2-T_0)(T_2-T_1)}\alpha(T_2)$$

$$\alpha(80)=\frac{2(80)-(40-40)}{(80-40)(80+40)}\left(6.47\times10^{-6}\right)+\frac{2(80)-(80-40)}{(40-80)(40+40)}\left(6.24\times10^{-6}\right)$$

$$+\frac{2(80)-(80+40)}{(-40-80)(-40-40)}\left(5.72\times10^{-6}\right)$$

Example 3 Cont.

$$= 33.33333\times10^{-3}\left(6.47\times10^{-6}\right) - 37.5\times10^{-3}\left(6.24\times10^{-6}\right) + 4.1666667\times10^{-3}\left(5.72\times10^{-6}\right)$$

$$= 5.49\times10^{-9}\ in/in/{}^{o}F^{2}$$

b) $$\alpha(T) = \left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)\alpha(T_0) + \left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)\alpha(T_1) + \left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)\alpha(T_2)$$

$$\frac{d\alpha}{dT}(T) = \frac{2T-(T_1+T_2)}{(T_0-T_1)(T_0-T_2)}\alpha(T_0) + \frac{2T-(T_0+T_2)}{(T_1-T_0)(T_1-T_2)}\alpha(T_1) + \frac{2T-(T_0+T_1)}{(T_2-T_0)(T_2-T_1)}\alpha(T_2)$$

$$\alpha(-340) = \frac{2(-340)-(-280-340)}{(-200+280)(-200+340)}\left(4.30\times10^{-6}\right) + \frac{2(-340)-(-200-340)}{(-280+200)(-280+340)}\left(3.33\times10^{-6}\right)$$

$$+\frac{2(-340)-(-200-280)}{(-340+200)(-340+280)}\left(2.45\times10^{-6}\right)$$

Example 3 Cont.

$$= -5.35\times10^{-3}\left(4.30\times10^{-6}\right) - 29.1\times10^{-3}\left(3.33\times10^{-6}\right) - 23.8\times10^{-3}\left(2.45\times10^{-6}\right)$$

$$= 15.7\times10^{-9}\, in/in/{}^{o}F^{2}$$