

## Chapter 02.03

# Differentiation of Discrete Functions

*After reading this chapter, you should be able to:*

- 1. find approximate values of the first derivative of functions that are given at discrete data points, and*
- 2. use Lagrange polynomial interpolation to find derivatives of discrete functions.*

To find the derivatives of functions that are given at discrete points, several methods are available. Although these methods are mainly used when the data is spaced unequally, they can be used for data that is spaced equally as well.

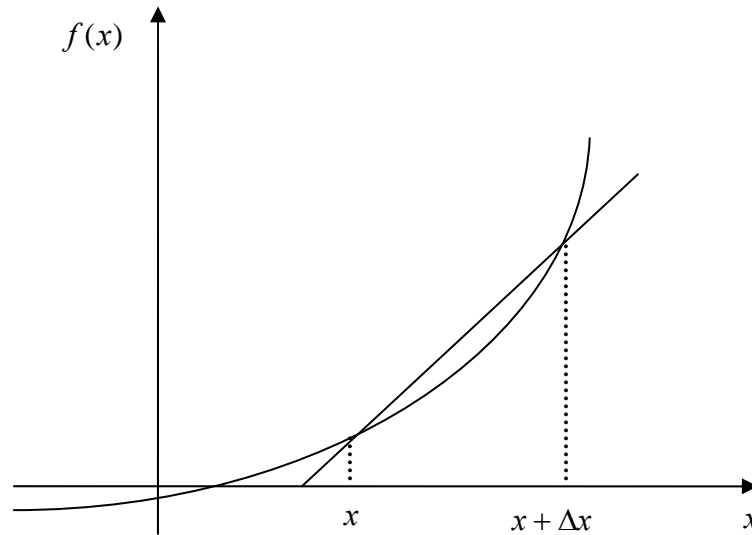
### **Forward Difference Approximation of the First Derivative**

We know

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite  $\Delta x$ ,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



**Figure 1** Graphical representation of forward difference approximation of first derivative.

So given  $n+1$  data points  $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ , the value of  $f'(x)$  for  $x_i \leq x \leq x_{i+1}$ ,  $i = 0, \dots, n-1$ , is given by

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

### Example 1

To find the contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 1.

**Table 1** Coefficient of thermal expansion as a function of temperature.

Temperature, $T$ (°F)	Coefficient of thermal expansion, $\alpha$ (in/in/°F)
80	$6.47 \times 10^{-6}$
40	$6.24 \times 10^{-6}$
-40	$5.72 \times 10^{-6}$
-120	$5.09 \times 10^{-6}$
-200	$4.30 \times 10^{-6}$
-280	$3.33 \times 10^{-6}$
-340	$2.45 \times 10^{-6}$

- (a) Is the rate of change of the coefficient of thermal expansion with respect to temperature more at  $T = 80^\circ\text{F}$  than at  $T = -340^\circ\text{F}$ ?

- (b) The data given in Table 1 can be regressed to  $\alpha = a_0 + a_1T + a_2T^2$  to get  
 $\alpha = 6.0216 \times 10^{-6} + 6.2790 \times 10^{-9}T - 1.2215 \times 10^{-11}T^2$ . Compare the results with part  
 (a) if you used the regression curve to find the rate of change of the coefficient of  
 thermal expansion with respect to temperature at  $T = 80^\circ\text{F}$  and at  $T = -340^\circ\text{F}$ .

### Solution

- (a) Using the forward divided difference approximation method at  $T = 80^\circ\text{F}$ ,

$$\begin{aligned}\frac{d\alpha(T_i)}{dT} &\approx \frac{\alpha(T_{i+1}) - \alpha(T_i)}{\Delta T} \\ T_i &= 80 \\ \Delta T &= -40 \\ T_{i+1} &= T_i + \Delta T \\ &= 80 + (-40) \\ &= 40 \\ \frac{d\alpha(80)}{dT} &\approx \frac{\alpha(40) - \alpha(80)}{-40} \\ &= \frac{6.24 \times 10^{-6} - 6.47 \times 10^{-6}}{-40} \\ &= 5.75 \times 10^{-9} \text{ in/in/}^\circ\text{F}^2\end{aligned}$$

Using the backward divided difference approximation method at  $T = -340^\circ\text{F}$ ,

$$\begin{aligned}\frac{d\alpha(T_i)}{dT} &\approx \frac{\alpha(T_i) - \alpha(T_{i-1})}{\Delta T} \\ T_i &= -340 \\ \Delta T &= -60 \\ T_{i-1} &= T_i - \Delta T \\ &= -340 - (-60) \\ &= -280 \\ \frac{d\alpha(-340)}{dT} &\approx \frac{\alpha(-340) - \alpha(-280)}{-60} \\ &= \frac{2.45 \times 10^{-6} - 3.33 \times 10^{-6}}{-60} \\ &= 0.14667 \times 10^{-7} \text{ in/in/}^\circ\text{F}^2\end{aligned}$$

From the above two results it is clear that the rate of change of the coefficient of thermal expansion is more at  $T = 80^\circ\text{F}$  than  $T = -340^\circ\text{F}$ .

- b) Given:

$$\begin{aligned}\alpha &= 6.0216 \times 10^{-6} + 6.2790 \times 10^{-9}T - 1.2215 \times 10^{-11}T^2 \\ \frac{d\alpha}{dT} &= 6.279 \times 10^{-9} - 2.443 \times 10^{-11}T\end{aligned}$$

$$\begin{aligned}\frac{d\alpha(80)}{dT} &= 6.279 \times 10^{-9} - 2.443 \times 10^{-11}(80) \\ &= 4.3246 \times 10^{-9} \text{ in/in/}^\circ\text{F}^2\end{aligned}$$

$$\begin{aligned}\frac{d\alpha(-340)}{dT} &= 6.279 \times 10^{-9} - 2.443 \times 10^{-11}(-340) \\ &= 0.14585 \times 10^{-7} \text{ in/in/}^\circ\text{F}^2\end{aligned}$$

**Table 2** Summary of change in coefficient of thermal expansion using different approximations.

Temperature, $T_i$	Change in Coefficient of Thermal Expansion, $\frac{d\alpha}{dT}(T_i)$	
	Divided Difference Approximation	2 <sup>nd</sup> Order Polynomial Regression
80	$5.75 \times 10^{-9} \text{ in/in/}^\circ\text{F}^2$	$4.3246 \times 10^{-9} \text{ in/in/}^\circ\text{F}^2$
-340	$0.14667 \times 10^{-7} \text{ in/in/}^\circ\text{F}^2$	$0.14585 \times 10^{-7} \text{ in/in/}^\circ\text{F}^2$

### Direct Fit Polynomials

In this method, given  $n+1$  data points  $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ , one can fit a  $n^{\text{th}}$  order polynomial given by

$$P_n(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n$$

To find the first derivative,

$$P'_n(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2x + \dots + (n-1)a_{n-1}x^{n-2} + na_nx^{n-1}$$

Similarly, other derivatives can also be found.

### Example 2

To find the contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 3.

**Table 3** Coefficient of thermal expansion as a function of temperature.

Temperature, $T$ ( $^\circ\text{F}$ )	Coefficient of thermal expansion, $\alpha$ (in/in/ $^\circ\text{F}$ )
80	$6.47 \times 10^{-6}$
40	$6.24 \times 10^{-6}$
-40	$5.72 \times 10^{-6}$
-120	$5.09 \times 10^{-6}$
-200	$4.30 \times 10^{-6}$
-280	$3.33 \times 10^{-6}$
-340	$2.45 \times 10^{-6}$

- (a) Using a third order polynomial interpolant, find the change in the coefficient of thermal expansion at  $T = 80^\circ\text{F}$  and  $T = -340^\circ\text{F}$ .
- (b) The data given in Table 3 can be regressed to  $\alpha = a_0 + a_1T + a_2T^2$  to get  $\alpha = 6.0216 \times 10^{-6} + 6.2790 \times 10^{-9}T - 1.2215 \times 10^{-11}T^2$ . Compare the results with part (a) if you used the regression curve to find the rate of change of the coefficient of thermal expansion with respect to temperature at  $T = 80^\circ\text{F}$  and  $T = -340^\circ\text{F}$ .

### Solution

For third order polynomial interpolation (also called cubic interpolation), we choose the coefficient of thermal expansion given by

$$\alpha(t) = a_0 + a_1T + a_2T^2 + a_3T^3$$

- (a) Change in the thermal expansion coefficient at  $80^\circ\text{F}$ :

Since we want to find the rate of change in the thermal expansion coefficient at  $T = 80^\circ\text{F}$ , and we are using a third order polynomial, we need to choose the four points closest to  $T = 80^\circ\text{F}$  that also bracket  $T = 80^\circ\text{F}$  to evaluate it.

The four points are  $T_0 = 80^\circ\text{F}$ ,  $T_1 = 40^\circ\text{F}$ ,  $T_2 = -40^\circ\text{F}$ , and  $T_3 = -120^\circ\text{F}$ .

$$T_0 = 80, \quad \alpha(T_0) = 6.47 \times 10^{-6}$$

$$T_1 = 40, \quad \alpha(T_1) = 6.24 \times 10^{-6}$$

$$T_2 = -40, \quad \alpha(T_2) = 5.72 \times 10^{-6}$$

$$T_3 = -120, \quad \alpha(T_3) = 5.09 \times 10^{-6}$$

such that

$$\alpha(80) = 6.47 \times 10^{-6} = a_0 + a_1(80) + a_2(80)^2 + a_3(80)^3$$

$$\alpha(40) = 6.24 \times 10^{-6} = a_0 + a_1(40) + a_2(40)^2 + a_3(40)^3$$

$$\alpha(-40) = 5.72 \times 10^{-6} = a_0 + a_1(-40) + a_2(-40)^2 + a_3(-40)^3$$

$$\alpha(-120) = 5.09 \times 10^{-6} = a_0 + a_1(-120) + a_2(-120)^2 + a_3(-120)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 80 & 6400 & 512000 \\ 1 & 40 & 1600 & 64000 \\ 1 & -40 & 1600 & -64000 \\ 1 & -120 & 14400 & -1728000 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 6.47 \times 10^{-6} \\ 6.24 \times 10^{-6} \\ 5.72 \times 10^{-6} \\ 5.09 \times 10^{-6} \end{bmatrix}$$

Solving the above gives

$$a_0 = 0.59915 \times 10^{-5}$$

$$a_1 = 0.64813 \times 10^{-8}$$

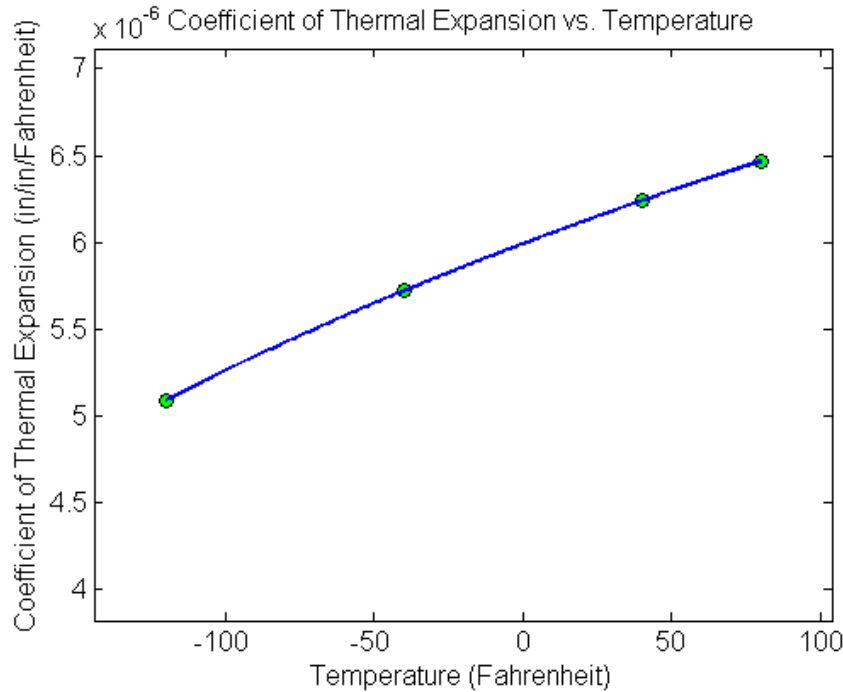
$$a_2 = -0.71875 \times 10^{-11}$$

$$a_3 = 0.11719 \times 10^{-13}$$

Hence

$$\alpha(T) = a_0 + a_1T + a_2T^2 + a_3T^3$$

$$= 0.59915 \times 10^{-5} + 0.64813 \times 10^{-8} T - 0.71875 \times 10^{-11} T^2 + 0.11719 \times 10^{-13} T^3, \quad -120 \leq T \leq 80$$



**Figure 2** Graph of coefficient of thermal expansion vs. temperature.

The change in the coefficient of thermal expansion at  $T = 80^\circ\text{F}$  is given by

$$\frac{d\alpha(80)}{dT} = \frac{d}{dT} \alpha(T) \Big|_{T=80}$$

Given that

$$\alpha(T) = 0.59915 \times 10^{-5} + 0.64813 \times 10^{-8} T - 0.71875 \times 10^{-11} T^2 + 0.11719 \times 10^{-13} T^3, \quad -120 \leq T \leq 80$$

$$\frac{d\alpha(T)}{dT} = \frac{d}{dT} \alpha(T)$$

$$= \frac{d}{dT} \left( \begin{array}{l} 0.59915 \times 10^{-5} + 0.64813 \times 10^{-8} T - 0.71875 \times 10^{-11} T^2 \\ + 0.11719 \times 10^{-13} T^3 \end{array} \right)$$

$$= 0.64813 \times 10^{-8} - 1.4375 \times 10^{-11} T + 0.35157 \times 10^{-13} T^2, \quad -120 \leq T \leq 80$$

$$\frac{d\alpha(80)}{dT} = 0.64812 \times 10^{-8} - 1.4375 \times 10^{-11} (80) + 0.35157 \times 10^{-13} (80)^2$$

$$= 5.5563 \times 10^{-9} \text{ in/in/}^\circ\text{F}^2$$

(b) Change in thermal expansion coefficient at  $-340^\circ\text{F}$ :

Since we want to find the rate of change in the thermal expansion coefficient at  $T = -340^\circ\text{F}$ , and we are using a third order polynomial, we need to choose the four points closest to  $T = -340^\circ\text{F}$  that also bracket  $T = -340^\circ\text{F}$  to evaluate it.

The four points are  $T_0 = -120^\circ\text{F}$ ,  $T_1 = -200^\circ\text{F}$ ,  $T_2 = -280^\circ\text{F}$ , and  $T_3 = -340^\circ\text{F}$ .

$$T_0 = -120, \quad \alpha(T_0) = 5.09 \times 10^{-6}$$

$$T_1 = -200, \quad \alpha(T_1) = 4.30 \times 10^{-6}$$

$$T_2 = -280, \quad \alpha(T_2) = 3.33 \times 10^{-6}$$

$$T_3 = -340, \quad \alpha(T_3) = 2.45 \times 10^{-6}$$

such that

$$\alpha(-120) = 5.09 \times 10^{-6} = a_0 + a_1(-120) + a_2(-120)^2 + a_3(-120)^3$$

$$\alpha(-200) = 4.30 \times 10^{-6} = a_0 + a_1(-200) + a_2(-200)^2 + a_3(-200)^3$$

$$\alpha(-280) = 3.33 \times 10^{-6} = a_0 + a_1(-280) + a_2(-280)^2 + a_3(-280)^3$$

$$\alpha(-340) = 2.45 \times 10^{-6} = a_0 + a_1(-340) + a_2(-340)^2 + a_3(-340)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & -120 & 14400 & -1728000 \\ 1 & -200 & 40000 & -8000000 \\ 1 & -280 & 78400 & -21952000 \\ 1 & -340 & 115600 & -39304000 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5.09 \times 10^{-6} \\ 4.30 \times 10^{-6} \\ 3.33 \times 10^{-6} \\ 2.45 \times 10^{-6} \end{bmatrix}$$

Solving the above gives

$$a_0 = 0.60625 \times 10^{-5}$$

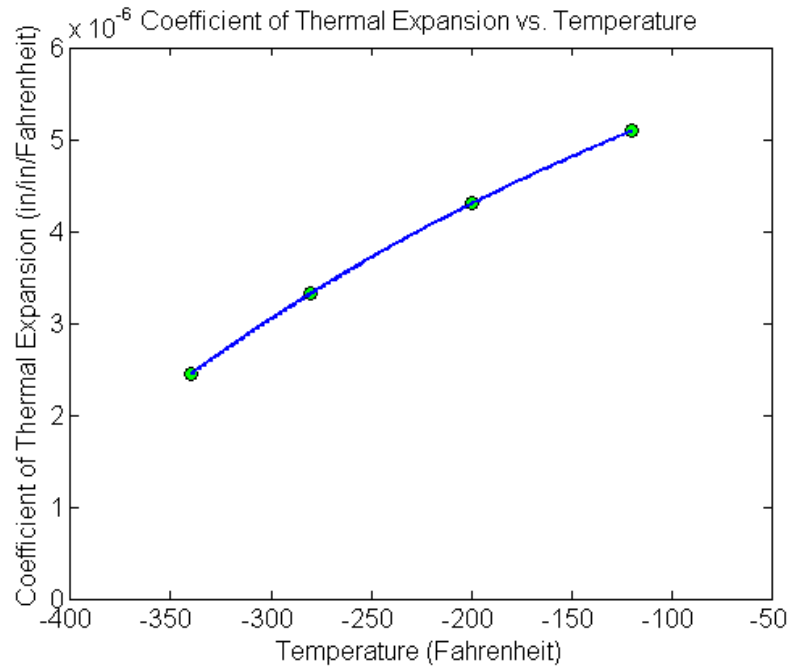
$$a_1 = 0.74881 \times 10^{-8}$$

$$a_2 = -0.29018 \times 10^{-11}$$

$$a_3 = 0.18601 \times 10^{-13}$$

Hence

$$\begin{aligned} \alpha(T) &= a_0 + a_1T + a_2T^2 + a_3T^3 \\ &= 0.60625 \times 10^{-5} + 0.74881 \times 10^{-8}T - 0.29018 \times 10^{-11}T^2 \\ &\quad + 0.18601 \times 10^{-13}T^3, \quad -340 \leq T \leq -120 \end{aligned}$$



**Figure 3** Graph of coefficient of thermal expansion vs. temperature.

The change in the coefficient of thermal expansion at  $T = -340$  °F is given by

$$\frac{d\alpha(-340)}{dT} = \frac{d}{dT} \alpha(T) \Big|_{T=-340}$$

Given that

$$\alpha(T) = 0.60625 \times 10^{-5} + 0.74881 \times 10^{-8} T - 0.29018 \times 10^{-11} T^2 + 0.18601 \times 10^{-13} T^3, \quad -340 \leq T \leq -120$$

$$\frac{d\alpha(T)}{dT} = \frac{d}{dT} \alpha(T)$$

$$= \frac{d}{dT} \left( \begin{array}{l} 0.60625 \times 10^{-5} + 0.74881 \times 10^{-8} T - 0.29018 \times 10^{-11} T^2 \\ + 0.18601 \times 10^{-13} T^3 \end{array} \right)$$

$$= 0.74881 \times 10^{-8} - 0.58036 \times 10^{-11} T + 0.55804 \times 10^{-13} T^2, \quad -340 \leq t \leq -120$$

$$\frac{d\alpha(-340)}{dT} = 0.74881 \times 10^{-8} - 0.58036 \times 10^{-11} (-340) + 0.55804 \times 10^{-13} (-340)^2$$

$$= 0.15905 \times 10^{-7} \text{ in/in/}^\circ\text{F}^2$$

**Table 4** Summary of change in coefficient of thermal expansion using different approximations.

Temperature, $T_i$	Change in Coefficient of Thermal Expansion, $\frac{d\alpha}{dT}(T_i)$	
	3 <sup>rd</sup> Order Interpolation	2 <sup>nd</sup> Order Polynomial Regression
80	$5.5563 \times 10^{-9} \text{ in/in/}^\circ\text{F}^2$	$4.3246 \times 10^{-9} \text{ in/in/}^\circ\text{F}^2$
-340	$0.15905 \times 10^{-7} \text{ in/in/}^\circ\text{F}^2$	$0.14585 \times 10^{-7} \text{ in/in/}^\circ\text{F}^2$

### Lagrange Polynomial

In this method, given  $(x_0, y_0), \dots, (x_n, y_n)$ , one can fit a  $n^{\text{th}}$  order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where  $n$  in  $f_n(x)$  stands for the  $n^{\text{th}}$  order polynomial that approximates the function  $y = f(x)$  and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$  is a weighting function that includes a product of  $n-1$  terms with terms of  $j=i$  omitted.

Then to find the first derivative, one can differentiate  $f_n(x)$  once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through  $(x_0, y_0), (x_1, y_1)$ , and  $(x_2, y_2)$  is

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating the above equation gives

$$f_2'(x) = \frac{2x - (x_1 + x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - (x_0 + x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2x - (x_0 + x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

### Example 3

To find the contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 5.

**Table 5** Coefficient of thermal expansion as a function of temperature.

Temperature, $T$ (°F)	Coefficient of thermal expansion, $\alpha$ (in/in/°F)
80	$6.47 \times 10^{-6}$
40	$6.24 \times 10^{-6}$
-40	$5.72 \times 10^{-6}$
-120	$5.09 \times 10^{-6}$
-200	$4.30 \times 10^{-6}$
-280	$3.33 \times 10^{-6}$
-340	$2.45 \times 10^{-6}$

- (a) Using a second order Lagrange polynomial interpolant, find the change in the coefficient of thermal expansion at  $T = 80^\circ\text{F}$  and  $T = -340^\circ\text{F}$ .
- (b) The data given in the Table 5 can be regressed to  $\alpha = a_0 + a_1T + a_2T^2$  to get  $\alpha = 6.0216 \times 10^{-6} + 6.2790 \times 10^{-9}T - 1.2215 \times 10^{-11}T^2$ . Compare the results with part (a) if you used the regression curve to find the rate of change of the coefficient of thermal expansion with respect to temperature at  $T = 80^\circ\text{F}$  and  $T = -340^\circ\text{F}$ .

**Solution**

For second order Lagrangian interpolation, we choose the coefficient of thermal expansion given by

$$\alpha(T) = \left( \frac{T-T_1}{T_0-T_1} \right) \left( \frac{T-T_2}{T_0-T_2} \right) \alpha(T_0) + \left( \frac{T-T_0}{T_1-T_0} \right) \left( \frac{T-T_2}{T_1-T_2} \right) \alpha(T_1) + \left( \frac{T-T_0}{T_2-T_0} \right) \left( \frac{T-T_1}{T_2-T_1} \right) \alpha(T_2)$$

- (a) Change in the thermal expansion coefficient at  $80^\circ\text{F}$ :

Since we want to find the rate of change in the thermal expansion coefficient at  $T = 80^\circ\text{F}$ , and we are using second order Lagrangian interpolation, we need to choose the three points closest to  $T = 80^\circ\text{F}$  that also bracket  $T = 80^\circ\text{F}$  to evaluate it.

The three points are  $T_0 = 80^\circ\text{F}$ ,  $T_1 = 40^\circ\text{F}$ , and  $T_2 = -40^\circ\text{F}$ .

$$T_0 = 80, \quad \alpha(T_0) = 6.47 \times 10^{-6}$$

$$T_1 = 40, \quad \alpha(T_1) = 6.24 \times 10^{-6}$$

$$T_2 = -40, \quad \alpha(T_2) = 5.72 \times 10^{-6}$$

The change in the coefficient of thermal expansion at  $T = 80^\circ\text{F}$  is given by

$$\frac{d\alpha(80)}{dT} = \frac{d}{dT} \alpha(T) \Big|_{T=80}$$

Hence

$$\frac{d\alpha}{dT}(T) = \frac{2T - (T_1 + T_2)}{(T_0 - T_1)(T_0 - T_2)} \alpha(T_0) + \frac{2T - (T_0 + T_2)}{(T_1 - T_0)(T_1 - T_2)} \alpha(T_1) + \frac{2T - (T_0 + T_1)}{(T_2 - T_0)(T_2 - T_1)} \alpha(T_2)$$

$$\begin{aligned}
\frac{d\alpha}{dT}(80) &= \frac{2(80) - (40 + (-40))}{(80 - 40)(80 - (-40))} (6.47 \times 10^{-6}) + \frac{2(80) - (80 + (-40))}{(40 - 80)(40 - (-40))} (6.24 \times 10^{-6}) \\
&\quad + \frac{2(80) - (80 + 40)}{(-40 - 80)(-40 - 40)} (5.72 \times 10^{-6}) \\
&= 2.1567 \times 10^{-7} - 2.34 \times 10^{-7} + 2.3833 \times 10^{-8} \\
&= 5.5 \times 10^{-9} \text{ in/in/}^\circ\text{F}^2
\end{aligned}$$

(b) Change in the thermal expansion coefficient at  $-340^\circ\text{F}$ :

Since we want to find the rate of change in the thermal expansion coefficient at  $T = -340^\circ\text{F}$ , and we are using second order Lagrangian interpolation, we need to choose the three points closest to  $T = -340^\circ\text{F}$  that also bracket  $T = -340^\circ\text{F}$  to evaluate it.

The three points are  $T_0 = -200^\circ\text{F}$ ,  $T_1 = -280^\circ\text{F}$ , and  $T_2 = -340^\circ\text{F}$ .

$$T_0 = -200, \alpha(T_0) = 4.30 \times 10^{-6}$$

$$T_1 = -280, \alpha(T_1) = 3.33 \times 10^{-6}$$

$$T_2 = -340, \alpha(T_2) = 2.45 \times 10^{-6}$$

The change in the coefficient of thermal expansion at  $T = -340^\circ\text{F}$  is given by

$$\frac{d\alpha(-340)}{dT} = \frac{d}{dT} \alpha(T) \Big|_{T=-340}$$

Hence

$$\frac{d\alpha}{dT}(T) = \frac{2T - (T_1 + T_2)}{(T_0 - T_1)(T_0 - T_2)} \alpha(T_0) + \frac{2T - (T_0 + T_2)}{(T_1 - T_0)(T_1 - T_2)} \alpha(T_1) + \frac{2T - (T_0 + T_1)}{(T_2 - T_0)(T_2 - T_1)} \alpha(T_2)$$

$$\begin{aligned}
\frac{d\alpha}{dT}(-340) &= \frac{2(-340) - (-280 + (-340))}{(200 - (-280))(-200 - (-340))} (4.30 \times 10^{-6}) \\
&\quad + \frac{2(-340) - (-200 + (-280))}{(-280 - (-200))(-280 - (-340))} (3.33 \times 10^{-6}) \\
&\quad + \frac{2(-340) - (-200 + (-280))}{(-340 - (-200))(-340 - (-280))} (2.45 \times 10^{-6}) \\
&= -2.3036 \times 10^{-8} + 9.7125 \times 10^{-8} - 5.8333 \times 10^{-8} \\
&= 0.15756 \times 10^{-7} \text{ in/in/}^\circ\text{F}^2
\end{aligned}$$

**Table 6** Summary of change in coefficient of thermal expansion using different approximations.

Temperature, $T_i$	Change in Coefficient of Thermal Expansion, $\frac{d\alpha}{dT}(T_i)$	
	2 <sup>nd</sup> Order Lagrange Interpolation	2 <sup>nd</sup> Order Polynomial Regression
80	$5.5 \times 10^{-9} \text{ in/in/}^\circ\text{F}^2$	$4.3246 \times 10^{-9} \text{ in/in/}^\circ\text{F}^2$
-340	$0.15756 \times 10^{-7} \text{ in/in/}^\circ\text{F}^2$	$0.14585 \times 10^{-7} \text{ in/in/}^\circ\text{F}^2$

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**DIFFERENTIATION**

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Topic	Differentiation of Discrete Functions
Summary	These are textbook notes differentiation of discrete functions
Major	Mechanical Engineering
Authors	Autar Kaw, Luke Snyder
Date	June 21, 2009
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>

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