Chapter 04.00G

Physical Problem for Simultaneous Linear Equations Mechanical Engineering

Problem Statement

To make the fulcrum (Figure 1) of a bascule bridge, a long hollow steel shaft called the trunnion is shrink fit into a steel hub. The resulting steel trunnion-hub assembly is then shrink fit into the girder of the bridge.

Figure 1 Trunnion-Hub-Girder (THG) assembly.

This is done by first immersing the trunnion in a cold medium such as dry-ice/alcohol mixture. After the trunnion reaches the steady state temperature of the cold medium, the trunnion outer diameter contracts. The trunnion is taken out of the medium and slid though the hole of the hub (Figure 2).

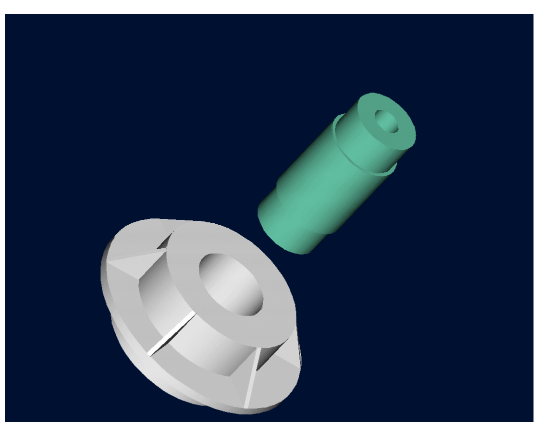

Figure 2 Trunnion slid through the hub after contracting

When the trunnion heats up, it expands and creates an interference fit with the hub. In 1995, on one of the bridges in Florida, this assembly procedure did not work as designed. Before the trunnion could be inserted fully into the hub, the trunnion got stuck. Luckily the trunnion was taken out before it got stuck permanently. Otherwise, a new trunnion and hub would needed to be ordered at a cost of $50,000. Coupled with construction delays, the total loss could have been more than hundred thousand dollars.

Why did the trunnion get stuck? This was because the trunnion had not contracted enough to slide through the hole. Can you find out why?

A hollow trunnion of outside diameter $12.363''$ is to be fitted in a hub of inner diameter $12.358''$. The trunnion was put in dry ice/alcohol mixture (temperature of the fluid - dry ice/alcohol mixture is $-108°\text{F}$) to contract the trunnion so that it can be slided through the hole of the hub. To slide the trunnion without sticking, a diametrical clearance of at least $0.01''$ is required between the trunnion and the hub. Assuming the room temperature is $80°\text{F}$, is immersing it in dry-ice/alcohol mixture a correct decision?

Solution

To calculate the contraction in the diameter of the trunnion, thermal expansion coefficient at room temperature is used. In that case the reduction, ΔD in the outer diameter of the trunnion is

$$\Delta D = D\alpha\Delta T \qquad (1)$$

where

D = outer diameter of the trunnion,

α = coefficient of thermal expansion coefficient at room temperature, and

ΔT = change in temperature,

Given

$D = 12.363''$

$\alpha = 6.817\times10^{-6}\ \text{in/in/°F}$ at $80°\text{F}$

$$\begin{aligned}\Delta T &= T_{fluid} - T_{room}\\ &= -108 - 80\\ &= -188°\text{F}\end{aligned}$$

where

T_{fluid} = temperature of dry-ice/alcohol mixture

T_{room} = room temperature

the reduction in the trunnion outer diameter is given by

$$\begin{aligned}\Delta D &= (12.363)\left(6.47\times10^{-6}\right)\left(-188\right)\\ &= -0.01504''\end{aligned}$$

So the trunnion is predicted to reduce in diameter by $0.01504''$. But, is this enough reduction in diameter? As per specifications, he needs the trunnion to contract by

= trunnion outside diameter - hub inner diameter + diametral clearance

$= 12.363'' - 12.358'' + 0.01''$

$= 0.015''$

So according to his calculations, immersing the steel trunnion in dry-ice/alcohol mixture gives the desired contraction of $0.015''$ as we predict a contraction of $0.01504''$.

But as shown in Figure 3, the thermal expansion coefficient of steel decreases with temperature and is not constant over the range of temperature the trunnion goes through. Hence, Equation 1 would overestimate the thermal contraction.

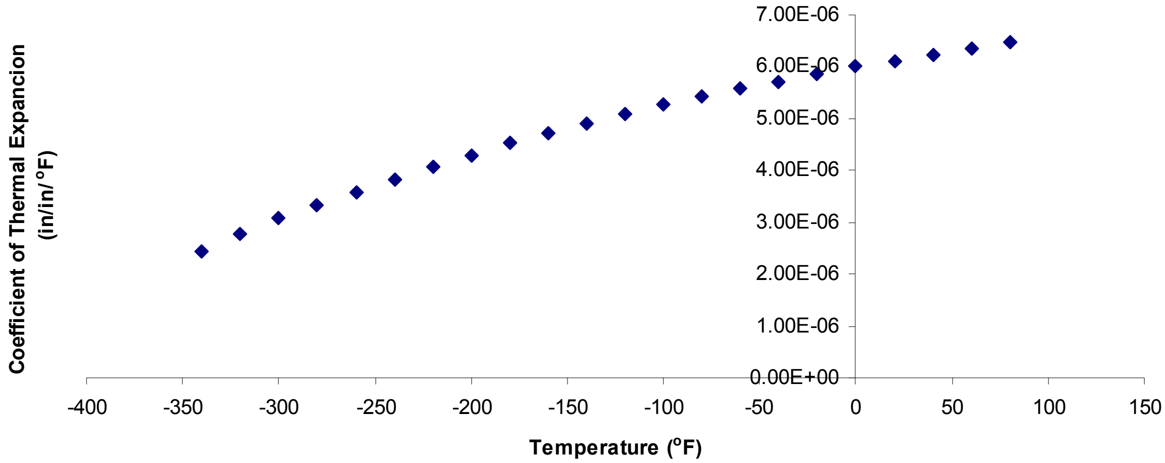

Figure 3 Varying thermal expansion coefficient as a function of temperature for cast steel.

The contraction in the diameter for the trunnion for which the thermal expansion coefficient varies as a function of temperature is given by

$$\Delta D = D\int_{T_{room}}^{T_{fluid}} \alpha dT \qquad (2)$$

So one needs to find the curve to find the coefficient of thermal expansion as a function of temperature. This is done by regression where we best fit a curve through the data given in Table 1.

Table 1 Instantaneous thermal expansion coefficient as a function of temperature.

Temperature	Instantaneous Thermal Expansion Coefficient
°F	μin/in/°F
80	6.47
60	6.36
40	6.24
20	6.12
0	6.00
-20	5.86
-40	5.72
-60	5.58
-80	5.43
-100	5.28
-120	5.09
-140	4.91
-160	4.72
-180	4.52

-200	4.30
-220	4.08
-240	3.83
-260	3.58
-280	3.33
-300	3.07
-320	2.76
-340	2.45

Assuming that the coefficient of thermal expansion is related to temperature by a second order polynomial,

$$\alpha = a_0 + a_1 T + a_2 T^2 \quad (3)$$

Given the data points (α_1, T_1), (α_2, T_2), …..,(α_n, T_n) as in Figure 3 and Table 1, the sum of the square of the residuals (sum of the square of the differences between the observed and predicted values) is

$$S_r = \sum_{i=1}^{n} \left(\alpha_i - \{a_0 + a_1 T_i + a_2 T_i^2\}\right)^2$$

$$= \sum_{i=1}^{n} \left(\alpha_i - a_0 - a_1 T_i - a_2 T_i^2\right)^2 \quad (4)$$

To minimize the value of the sum of the square of the residuals, we take the derivative with respect to each of the three unknown coefficients to give

$$\frac{\partial S_r}{\partial a_0} = \sum_{i=1}^{n} 2\left(\alpha_i - a_0 - a_1 T_i - a_2 T_i^2\right)(-1)$$

$$= 2\left[-\sum_{i=1}^{n} \alpha_i + n a_0 + a_1 \sum_{i=1}^{n} T_i + a_2 \sum_{i=1}^{n} T_i^2\right]$$

$$\frac{\partial S_r}{\partial a_1} = \sum_{i=1}^{n} 2\left(\alpha_i - a_0 - a_1 T_i - a_2 T_i^2\right)(-T_i)$$

$$= 2\left[-\sum_{i=1}^{n} \alpha_i T_i + a_0 \sum_{i=1}^{n} T_i + a_1 \sum_{i=1}^{n} T_i^2 + a_2 \sum_{i=1}^{n} T_i^3\right]$$

$$\frac{\partial S_r}{\partial a_2} = \sum_{i=1}^{n} 2\left(\alpha_i - a_0 - a_1 T_i - a_2 T_i^2\right)(-T_i^2)$$

$$= 2\left[-\sum_{i=1}^{n} \alpha_i T_i^2 + a_0 \sum_{i=1}^{n} T_i^2 + a_1 \sum_{i=1}^{n} T_i^3 + a_2 \sum_{i=1}^{n} T_i^4\right] \quad (5)$$

Setting three partial derivatives in Equation (5) equal to zero gives,

$$n a_0 + a_1 \sum_{i=1}^{n} T_i + a_2 \sum_{i=1}^{n} T_i^2 = \sum_{i=1}^{n} \alpha_i$$

$$a_0 \sum_{i=1}^{n} T_i + a_1 \sum_{i=1}^{n} T_i^2 + a_2 \sum_{i=1}^{n} T_i^3 = \sum_{i=1}^{n} \alpha_i T_i$$

$$a_0 \sum_{i=1}^{n} T_i^2 + a_1 \sum_{i=1}^{n} T_i^3 + a_2 \sum_{i=1}^{n} T_i^4 = \sum_{i=1}^{n} \alpha_i T_i^2 \quad (6)$$

The set of equations given by equations(6a), (6b), and (6c) are simultaneous linear equations. The number of data points in the Figure (3) is 24 as given in Table 1. Hence

$$n = 24$$

$$\sum_{i=1}^{24} T_i = -2860$$

$$\sum_{i=1}^{24} T_i^2 = 7.26 \times 10^5$$

$$\sum_{i=1}^{24} T_i^3 = -1.86472 \times 10^8$$

$$\sum_{i=1}^{24} T_i^4 = 5.24357 \times 10^{10}$$

$$\sum_{i=1}^{24} \alpha_i = 1.057 \times 10^{-4}$$

$$\sum_{i=1}^{24} \alpha_i T_i = -1.04162 \times 10^{-2}$$

$$\sum_{i=1}^{24} \alpha_i T_i^2 = 2.56799$$

$$24a_0 - 2860a_1 + 7.26 \times 10^5 a_2 = 1.057 \times 10^{-4}$$

$$-2860a_0 + 7.26 \times 10^5 a_1 - 1.8647 \times 10^8 a_2 = -1.04162 \times 10^{-2}$$

$$7.26 \times 10^5 a_0 - 1.86472 \times 10^8 a_1 + 5.24357 \times 10^{10} a_2 = 2.56799$$

In matrix form, the three simultaneous linear equations can be written as

$$\begin{bmatrix} 24 & -2860 & 7.26 \times 10^5 \\ -2860 & 7.26 \times 10^5 & -1.86472 \times 10^8 \\ 7.26 \times 10^5 & -1.86472 \times 10^8 & 5.24357 \times 10^{10} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1.057 \times 10^{-4} \\ -1.04162 \times 10^{-2} \\ 2.56799 \end{bmatrix}$$

QUESTIONS

1. Can you now find the contraction in the trunnion outer diameter?
2. Is the magnitude of contraction more than $0.015''$ as required?
3. If that is not the case, what if the trunnion were immersed in liquid nitrogen (boiling temperature= $-321°\text{F}$)? Will that give enough contraction in the trunnion?
4. Redo problem#1 using a third order polynomial as the regression model. How much different is the estimate of contraction using the third order polynomial?
5. Find the optimum polynomial order to use for the regression model.
6. Find the effect of the number of significant digits used in solving the set of the equations for problem#4 as you must have noticed the large range in the order of the numbers in the coefficient matrix.

SIMULTANEOUS LINEAR EQUATIONS	
Topic	Simultaneous linear equations
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