Simultaneous Linear Equations

Holistic Numerical Methods Institute

Topic: Gaussian Elimination

Major: Mechanical Engineering

Gaussian Elimination

One of the most popular techniques for solving simultaneous linear equations of the form $[A][X] = [C]$

Consists of 2 steps

1. Forward Elimination of Unknowns.

2. Back Substitution

Forward Elimination

The goal of Forward Elimination is to transform the coefficient matrix into an Upper Triangular Matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Forward Elimination

Linear Equations

A set of *n* equations and *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

Forward Elimination

Transform to an Upper Triangular Matrix

Step 1: Eliminate x_1 in 2^{nd} equation using equation 1 as the pivot equation

$$\left[\frac{Eqn1}{a_{11}}\right]\times(a_{21})$$

Which will yield

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Forward Elimination

Zeroing out the coefficient of x_1 in the 2nd equation.

Subtract this equation from 2nd equation

$$\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right) x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}\right) x_n = b_2 - \frac{a_{21}}{a_{11}} b_1$$

Or

$$a'_{22} x_2 + \ldots + a'_{2n} x_n = b'_2$$

Where

$$a'_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

$$\vdots$$

$$a'_{2n} = a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}$$

Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n &= b'_3 \\
\vdots \qquad \vdots \qquad \qquad \vdots \\
a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n &= b'_n
\end{aligned}
$$

Forward Elimination

Step 2: Eliminate x_2 in the 3^{rd} equation.

Equivalent to eliminating x_1 in the 2^{nd} equation using equation 2 as the pivot equation.

$$Eqn3 - \left[\frac{Eqn2}{a_{22}}\right] \times (a_{32})$$

Forward Elimination

This procedure is repeated for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a''_{n3}x_3 + \ldots + a''_{nn}x_n = b''_n$$

Forward Elimination

Continue this procedure by using the third equation as the pivot equation and so on.

At the end of (n-1) Forward Elimination steps, the system of equations will look like:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Forward Elimination

At the end of the Forward Elimination steps

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ & & a''_{33} & \cdots & a''_{3n} \\ & & & \vdots & \vdots \\ & & & & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Back Substitution

The goal of Back Substitution is to solve each of the equations using the upper triangular matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Example of a system of 3 equations

Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Solve the second from last equation $(n-1)^{th}$ using x_n solved for previously.

This solves for x_{n-1}.

Back Substitution

Representing Back Substitution for all equations by formula

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}$$

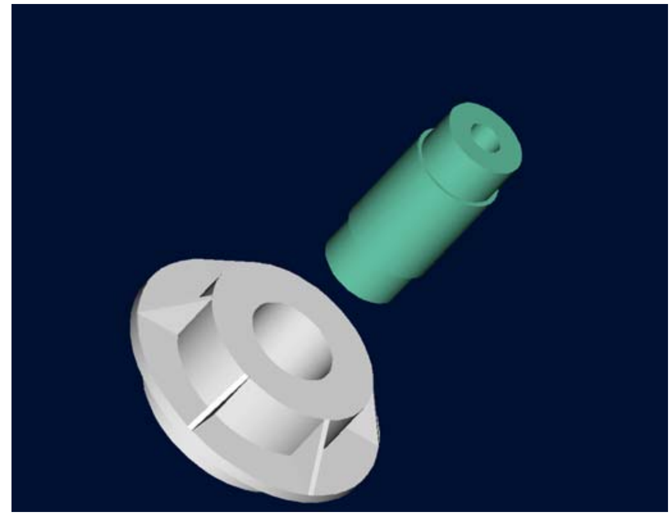
For $i=n-1, n-2, \ldots, 1$

and

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Example: Thermal Coefficient

A trunnion of diameter 12.363” has to be cooled from a room temperature of 80°F before it is shrink fit into a steel hub



The equation that gives the diametric contraction ΔD of the trunnion in dry-ice/alcohol (boiling temperature is -108°F is given by:

$$\Delta D = 12.363 \int_{80}^{-108} \alpha(T)\,dT$$

Example: Thermal Coefficient

The expression for the thermal expansion coefficient,

$\alpha = a_1 + a_2T + a_3T^2$ is obtained using regression analysis and hence solving the following simultaneous linear equations:

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

Find the values of a_1, a_2, and a_3 using Naïve Guass Elimination.

Example: Thermal Coefficient

Forward Elimination: Step 1

$$\text{Row}2 - \left[\frac{\text{Row}1}{24}\right]\times(-2860) =$$

Yields

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ 0 & 3.85183\times10^{5} & -99.957\times10^{6} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ 2.1797\times10^{-3} \\ 2.56799 \end{bmatrix}$$

Example: Thermal Coefficient

Forward Elimination: Step 1

$$\text{Row3} - \left[\frac{\text{Row1}}{24}\right] \times (7.26 \times 10^5) =$$

Yields

$$\begin{bmatrix} 24 & -2860 & 7.26 \times 10^5 \\ 0 & 3.85183 \times 10^5 & -99.957 \times 10^6 \\ 0 & -99.957 \times 10^6 & 30.4742 \times 10^9 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057 \times 10^{-4} \\ 2.1797 \times 10^{-3} \\ -0.62944 \end{bmatrix}$$

Example: Thermal Coefficient

Forward Elimination: Step 2

$$\text{Row3} - \left[\frac{\text{Row2}}{3.85183\times10^{5}}\right]\times(-99.957\times10^{6}) =$$

Yields

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ 0 & 3.85183\times10^{5} & -99.957\times10^{6} \\ 0 & 0 & 4.5348\times10^{9} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ 2.1797\times10^{-3} \\ -6.3797\times10^{-2} \end{bmatrix}$$

This is now ready for Back Substitution

Example: Thermal Coefficient

Back Substitution: Solve for a_3 using the third equation

$$4.5348\times10^{9}a_3=-6.3797\times10^{-2}$$

$$a_3=\frac{-6.3797\times10^{-2}}{4.5348\times10^{9}}$$

$$=-1.4068\times10^{-11}$$

Example: Thermal Coefficient

Back Substitution: Solve for a_2 using the second equation

$$3.85183\times10^{5}a_2+(-99.957\times10^{6})a_3=2.1797\times10^{-3}$$

$$a_2=\frac{2.1797\times10^{-3}-(-99.957\times10^{6})a_3}{3.85183\times10^{5}}$$

$$=\frac{2.1797\times10^{-3}-(-99.957\times10^{6})\times(-1.4068\times10^{-11})}{3.85183\times10^{5}}$$

$$=2.0081\times10^{-9}$$

Example: Thermal Coefficient

Back Substitution: Solve for a_1 using the first equation

$$24a_1 + (-2860)a_2 + 7.26\times10^5 a_3 = 1.057\times10^{-4}$$

$$a_1 = \frac{1.057\times10^{-4} - (-2860)a_2 - 7.26\times10^5 a_3}{24}$$

$$= \frac{1.057\times10^{-4} - (-2860)*(2.0081\times10^{-9}) - 7.26\times10^5 * (-1.4068\times10^{-11})}{24}$$

$$= 5.069\times10^{-6}$$

Example: Thermal Coefficient

Solution:

The solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5.069\times10^{-6} \\ 2.0081\times10^{-9} \\ -1.4068\times10^{-11} \end{bmatrix}$$

The polynomial that passes through the three data points is then:

$$\alpha(T) = a_1T^2 + a_2T + a_3$$

$$= 5.069\times10^{-6}T^2 + 2.0081\times10^{-9}T - 1.4068\times10^{-11}$$

Pitfalls

Two Potential Pitfalls

-Division by zero: May occur in the forward elimination steps. Consider the set of equations:

$$10x_2 - 7x_3 = 7$$

$$6x_1 + 2.099x_2 + 3x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

- Round-off error: Prone to round-off errors.

Pitfalls: Example

Consider the system of equations:

Use five significant figures with chopping

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

At the end of Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

Pitfalls: Example

Back Substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

$$x_3 = \frac{15004}{15005} = 0.99993$$

$$x_2 = \frac{6.001 - 6x_3}{-0.001} = -1.5$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = -0.3500$$

Pitfalls: Example

Compare the calculated values with the exact solution

$$[X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$

Improvements

Increase the number of significant digits

- Decreases round off error
- Does not avoid division by zero

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

Partial Pivoting

Gaussian Elimination with partial pivoting applies row switching to normal Gaussian Elimination.

How?

At the beginning of the kth step of forward elimination, find the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \ldots\ldots\ldots\ldots\ldots, |a_{nk}|$$

If the maximum of the values is $|a_{pk}|$ In the pth row, $k \le p \le n,$

then switch rows p and k.

Partial Pivoting

What does it Mean?

Gaussian Elimination with Partial Pivoting ensures that each step of Forward Elimination is performed with the pivoting element $|a_{kk}|$ having the largest absolute value.

Partial Pivoting: Example

Consider the system of equations

$$
\begin{aligned}
&10x_1 - 7x_2 = 7 \\
&-3x_1 + 2.099x_2 + 3x_3 = 3.901 \\
&5x_1 - x_2 + 5x_3 = 6
\end{aligned}
$$

In matrix form

$$
\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}
$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

|10|, |-3|, and |5| or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

|-0.001| and |2.5| or 0.0001 and 2.5

The largest absolute value is 2.5, so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_2}{2.5} = 1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$

Partial Pivoting: Example

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad [X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Summary

-Forward Elimination

-Back Substitution

-Pitfalls

-Improvements

-Partial Pivoting