Simultaneous Linear Equations

Holistic Numerical Methods Institute

Topic: Gauss-Seidel Method

Major: Mechanical Engineering

Gauss-Seidel Method

An *iterative* method.

Basic Procedure:

-Algebraically solve each linear equation for x_i

-Assume an initial guess solution array

-Solve for each x_i and repeat

-Use absolute relative approximate error after each iteration to check if error is within a pre-specified tolerance.

Gauss-Seidel Method

Why?

The Gauss-Seidel Method allows the user to control round-off error.

Elimination methods such as Gaussian Elimination and LU Decomposition are prone to prone to round-off error.

Also: If the physics of the problem are understood, a close initial guess can be made, decreasing the number of iterations needed.

Gauss-Seidel Method

Algorithm

A set of *n* equations and *n* unknowns:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

If: the diagonal elements are non-zero

Rewrite each equation solving for the corresponding unknown

ex:

First equation, solve for x_1

Second equation, solve for x_2

Gauss-Seidel Method

Algorithm

Rewriting each equation

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$ ← From Equation 1

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$ ← From equation 2

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$ ← From equation n-1

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$ ← From equation n

Gauss-Seidel Method

Algorithm

General Form of each equation

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j \neq 1}}^{n} a_{1j} x_j}{a_{11}}$$

$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j \neq n-1}}^{n} a_{n-1,j} x_j}{a_{n-1,n-1}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j \neq 2}}^{n} a_{2j} x_j}{a_{22}}$$

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j \neq n}}^{n} a_{nj} x_j}{a_{nn}}$$

Gauss-Seidel Method

Algorithm

General Form for any row 'i'

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, i = 1,2,\ldots,n.$$

How or where can this equation be used?

Gauss-Seidel Method

Solve for the unknowns

Assume an initial guess for [X]

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Use rewritten equations to solve for each value of xi.

Important: Remember to use the most recent value of x_i. Which means to apply values calculated to the calculations remaining in the **current** iteration.

Gauss-Seidel Method

Calculate the Absolute Relative Approximate Error

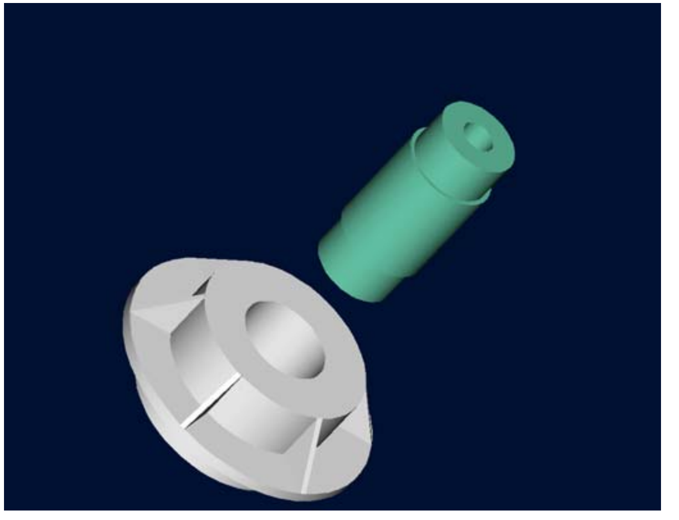
$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

So when has the answer been found?

The iterations are stopped when the absolute relative approximate error is less than a pre-specified tolerance for all unknowns.

Example: Thermal Coefficient

A trunnion of diameter 12.363" has to be cooled from a room temperature of 80°F before it is shrink fit into a steel hub



The equation that gives the diametric contraction ΔD of the trunnion in dry-ice/alcohol (boiling temperature is -108°F is given by:

$$\Delta D = 12.363 \int_{80}^{-108} \alpha(T)\,dT$$

Example: Thermal Coefficient

The expression for the thermal expansion coefficient,

$\alpha = a_1 + a_2T + a_3T^2$ is obtained using regression analysis and hence solving the following simultaneous linear equations:

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

Find the values of a_1, a_2, and a_3 using Gauss Seidel Method.

Example: Thermal Coefficient

The system of equations is:

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

Initial Guess: Assume an initial guess of $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example: Thermal Coefficient

Rewriting each equation

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

$$a_1 = \frac{1.057\times10^{-4} - (-2860)\times 0 - 7.26\times10^{5}\times 0}{24} = 4.4017\times10^{-6}$$

$$a_2 = \frac{-1.04162\times10^{-2} - (-2860)\times 4.40417\times10^{-6} - (-1.86472\times10^{8})\times 0}{7.26\times10^{5}} = 3.00236\times10^{5}$$

$$a_3 = \frac{2.56799 - 7.26\times10^{5}\times 4.40417\times10^{-6} - (-1.86472\times10^{8})\times 3.00236\times10^{-9}}{5.24357\times10^{10}} = -1.32692\times10^{-12}$$

Example: Thermal Coefficient

Finding the absolute relative approximate error

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

$$\left|\varepsilon_a\right|_1 = \left|\frac{4.40417\times10^{-6} - 0}{4.40417\times10^{-6}}\right| x100 = 100\%$$

$$\left|\varepsilon_a\right|_2 = \left|\frac{3.00236\times10^{-9} - 1000}{3.00236\times10^{-9}}\right| x100 = 100\%$$

$$\left|\varepsilon_a\right|_3 = \left|\frac{-1.32692\times10^{-12} - 100}{-1.32692\times10^{-12}}\right| x100 = 100\%$$

At the end of the first iteration

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4.40417\times10^{-6} \\ 3.00236\times10^{-9} \\ -1.32692\times10^{-12} \end{bmatrix}$$

The maximum absolute relative approximate error is 100.00%

Example: Thermal Coefficient

Iteration #2

Using $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4.40417\times10^{-6} \\ 3.00236\times10^{-9} \\ -1.32692\times10^{-12} \end{bmatrix}$ from iteration #1 the values of a_i are found:

$$a_1 = \frac{1.057\times10^{-4} - (-2860)\times3.00236\times10^{-9} - 7.26\times10^{5}\times(-1.32692\times10^{-12})}{24}$$

$$= 4.80209\times10^{-6}$$

$$a_2 = \frac{-1.04162\times10^{-2} - (-2860)\times4.80209\times10^{-6} - (-1.86472\times10^{8})\times(-1.32692\times10^{-12})}{7.26\times10^{5}}$$

$$= 4.22911\times10^{-9}$$

$$a_3 = \frac{2.56799 - 7.26\times10^{5}\times4.80209\times10^{-6} - (-1.86472\times10^{8})\times4.22911\times10^{-9}}{5.24357\times10^{10}}$$

$$= -2.47378\times10^{-12}$$

Example: Thermal Coefficient

Finding the absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{4.80209\times10^{-6} - 4.40417\times10^{-6}}{4.80209\times10^{-6}}\right| x100 = 8.28642\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.22911\times10^{-9} - 3.00236\times10^{-9}}{4.22911\times10^{-9}}\right| x100 = 29.00726\%$$

$$\left|\in_a\right|_3 = \left|\frac{-2.47378\times10^{-12} - (-1.32692\times10^{-12})}{-2.47378\times10^{-12}}\right| x100 = 46.36050\%$$

At the end of the second iteration

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4.80209\times10^{-6} \\ 4.22911\times10^{-9} \\ -2.47378\times10^{-12} \end{bmatrix}$$

The maximum absolute relative approximate error is 46.36050%

Example: Thermal Coefficient

Repeating more iterations, the following values are obtained

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	a_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
1	4.4042×10^{-6}	100	3.0024×10^{-9}	100	-1.3269×10^{-12}	100
2	4.8021×10^{-6}	8.2864	4.2291×10^{-9}	29.0073	-2.4767×10^{-12}	46.3605
3	4.9830×10^{-6}	3.6300	4.6471×10^{-9}	8.9946	-3.4917×10^{-12}	29.1527
4	5.0636×10^{-6}	1.5918	4.7023×10^{-9}	1.1922	-4.4083×10^{-12}	20.7922
5	5.0980×10^{-6}	0.6749	4.6033×10^{-9}	2.1696	-5.2397×10^{-12}	15.8702
6	5.1112×10^{-6}	0.2593	4.4419×10^{-9}	3.6330	-5.9972×10^{-12}	12.6290

! Notice – After six iterations, the absolute relative approximate errors are decreasing, but are still high.

Example: Thermal Coefficient

Repeating more iterations, the following values are obtained

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	a_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
75	$5.06916\text{x}10^{-6}$	2.25585x10-4	$2.0136\text{x}10^{-9}$	0.02428	$-1.4049\text{x}10^{-11}$	0.01125
76	$5.06915\text{x}10^{-6}$	2.0630x10-4	$2.0135\text{x}10^{-9}$	0.02221	$-1.4051\text{x}10^{-11}$	0.01029

The value of

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5.0691\times10^{-6} \\ 2.0135\times10^{-9} \\ -1.4051\times10^{-11} \end{bmatrix}$$

closely approaches the true value of

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5.06903\times10^{-6} \\ 2.00869\times10^{-9} \\ -1.406607\times10^{-11} \end{bmatrix}$$

Example: Thermal Coefficient

The polynomial that passes through the three data points is then

$$\alpha(T) = a_1 T^2 + a_2 T + a_3$$

$$= 5.06915 \times 10^{-6} T^2 + 2.00135 \times 10^{-9} T - 1.4051 \times 10^{-11}$$

Gauss-Seidel Method: Potential Pitfall

Diagonally dominant: The coefficient on the diagonal must be at least equal to the sum of the other coefficients in that row and at least one row with a diagonal coefficient greater than the sum of the other coefficients in that row.

Non-Diagonally dominant

$$[A] = \begin{bmatrix} 2 & 5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix}$$

Diagonally dominant

$$[B] = \begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 34 & 129 \end{bmatrix}$$

Most physical systems, such as the previous example, result in simultaneous linear equations that have diagonally dominant coefficient matrices

Gauss-Seidel Method

Summary

-Advantages of the Gauss-Seidel Method

-Algorithm for the Gauss-Seidel Method

-Pitfalls of the Gauss-Seidel Method

Gauss-Seidel Method

Questions?