

Interpolation

Topic: Direct Method

Major: Mechanical



What is Interpolation ?

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, find the value of 'y' at a value of 'x' that is not given.





Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.



Direct Method

Given ' $n+1$ ' data points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, pass a polynomial of order ' n ' through the data as given below:

$$y = a_0 + a_1x + \dots + a_nx^n.$$

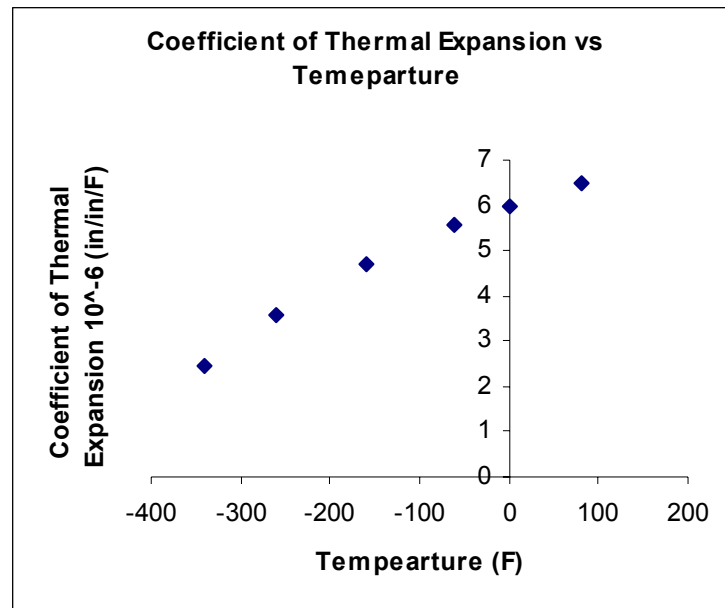
where a_0, a_1, \dots, a_n are real constants.

- Set up ' $n+1$ ' equations to find ' $n+1$ ' constants.
- To find the value ' y ' at a given value of ' x ', simply substitute the value of ' x ' in the above polynomial.

Example

A trunnion is cooled 80°F to -108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at $T = -14^\circ\text{F}$ using the direct method for linear interpolation.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
-60	5.58×10^{-6}
-160	4.72×10^{-6}
-260	3.58×10^{-6}
-340	2.45×10^{-6}



Linear Interpolation

$$\alpha(T) = a_0 + a_1 T$$

$$\alpha(0) = a_0 + a_1(0) = 6.00 \times 10^{-6}$$

$$\alpha(-60) = a_0 + a_1(-60) = 5.58 \times 10^{-6}$$

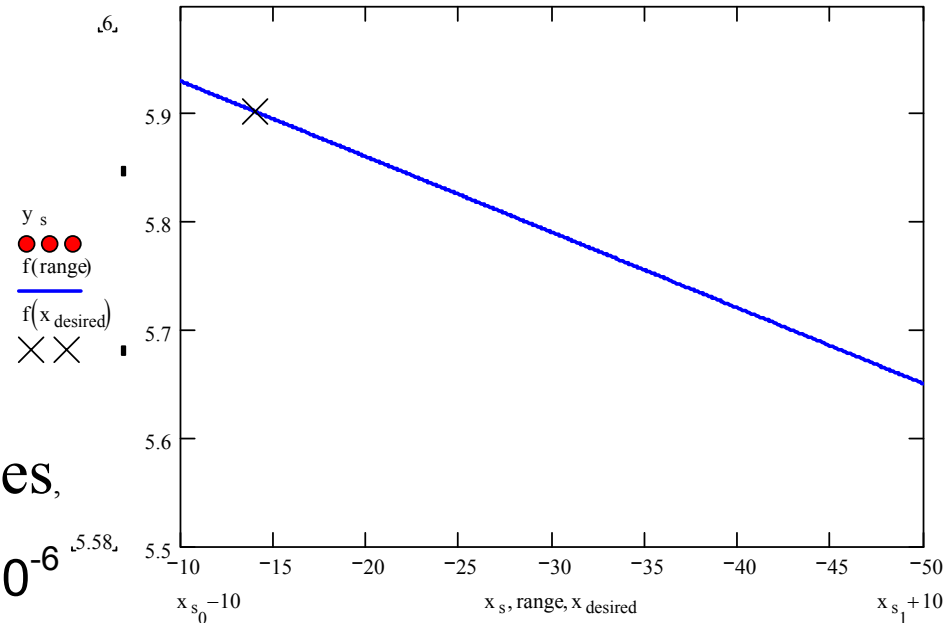
Solving the above two equations gives,

$$a_0 = 6.00 \times 10^{-6} \quad a_1 = 0.007 \times 10^{-6}$$

Hence

$$\alpha(T) = 6.00 \times 10^{-6} + 0.007 \times 10^{-6} T, \quad -60 \leq T \leq 0.$$

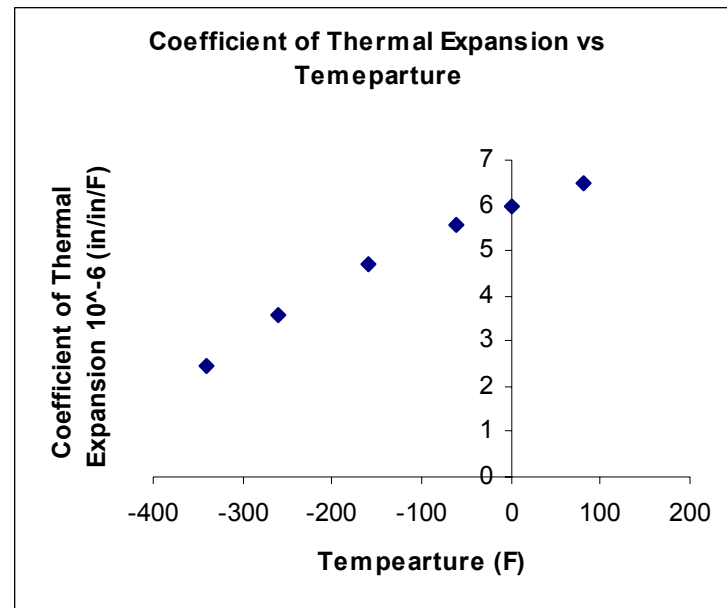
$$\alpha(-14) = 6.00 \times 10^{-6} + 0.007 \times 10^{-6}(-14) = 5.902 \times 10^{-6} \text{ in/in/}^\circ\text{F}$$



Example

A trunnion is cooled 80°F to -108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at $T = -14^\circ\text{F}$ using the direct method for quadratic interpolation.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
-60	5.58×10^{-6}
-160	4.72×10^{-6}
-260	3.58×10^{-6}
-340	2.45×10^{-6}





Quadratic Interpolation

$$\alpha(T) = a_0 + a_1T + a_2T^2$$

$$\alpha(80) = a_0 + a_1(80) + a_2(80)^2 = 6.47 \times 10^{-6}$$

$$\alpha(0) = a_0 + a_1(0) + a_2(0)^2 = 6.00 \times 10^{-6}$$

$$\alpha(-60) = a_0 + a_1(-60) + a_2(-60)^2 = 5.58 \times 10^{-6}$$

Solving the above three equations gives

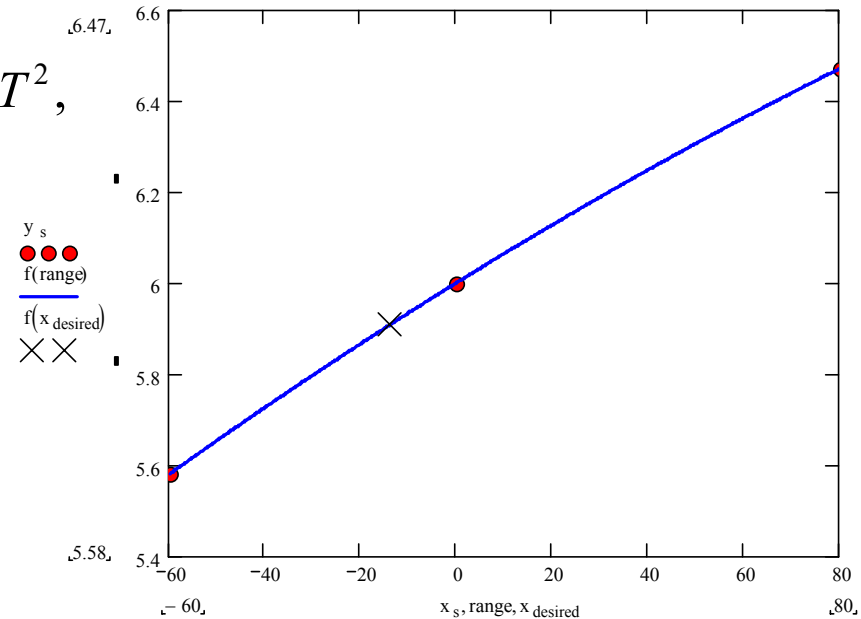
$$a_0 = 6.00 \times 10^{-6} \quad a_1 = 6.5179 \times 10^{-9} \quad a_2 = -8.0357 \times 10^{-12}$$

Quadratic Interpolation (contd)

$$\alpha(T) = 6.00 \times 10^{-6} + 6.5179 \times 10^{-9} T - 8.0357 \times 10^{-12} T^2, \\ -60 \leq T \leq 80$$

$$\alpha(-14) = 6.00 \times 10^{-6} + 6.5179 \times 10^{-9} (-14) \\ - 8.0357 \times 10^{-12} (-14)^2 \\ = 5.9072 \times 10^{-6}$$

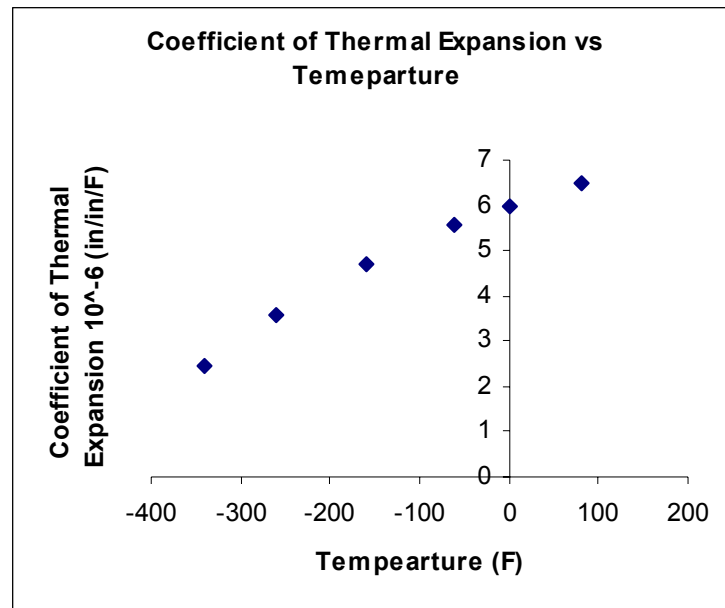
$$|\epsilon_a| = \left| \frac{5.9072 \times 10^{-6} - 5.902 \times 10^{-6}}{5.9072 \times 10^{-6}} \right| \times 100 = 0.08761\%$$



Example

A trunnion is cooled 80°F to -108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at $T = -14^\circ\text{F}$ using the direct method for cubic interpolation.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
-60	5.58×10^{-6}
-160	4.72×10^{-6}
-260	3.58×10^{-6}
-340	2.45×10^{-6}





Cubic Interpolation

$$\alpha(T) = a_0 + a_1T + a_2T^2 + a_3T^3$$

$$\alpha(80) = 6.47 \times 10^{-6} = a_0 + a_1(80) + a_2(80)^2 + a_3(80)^3$$

$$\alpha(0) = 6.00 \times 10^{-6} = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3$$

$$\alpha(-60) = 5.58 \times 10^{-6} = a_0 + a_1(-60) + a_2(-60)^2 + a_3(-60)^3$$

$$\alpha(-160) = 4.72 \times 10^{-6} = a_0 + a_1(-160) + a_2(-160)^2 + a_3(-160)^3$$

$$a_0 = 6.00 \times 10^{-6}$$

$$a_1 = 6.4786 \times 10^{-9}$$

$$a_2 = -8.1994 \times 10^{-12}$$

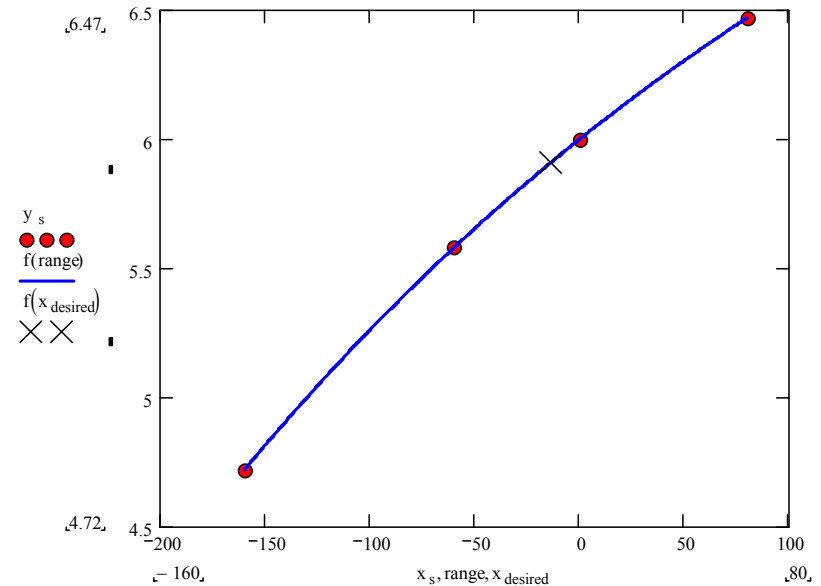
$$a_3 = 8.1845 \times 10^{-15}$$

Cubic Interpolation (contd)

$$\alpha(T) = 6.00 \times 10^{-6} + 6.4786 \times 10^{-9} T - 8.1994 \times 10^{-12} T^2 + 8.1845 \times 10^{-15} T^3, \quad -160 \leq T \leq 80,$$

$$\begin{aligned} \alpha(-14) &= 6.00 \times 10^{-6} + 6.4786 \times 10^{-9}(-14) - 8.1994 \times 10^{-12}(-14)^2 + 8.1845 \times 10^{-15}(-14)^3 \\ &= 5.9077 \times 10^{-6} \text{ in/in/}^\circ\text{F} \end{aligned}$$

$$\begin{aligned} |\epsilon_a| &= \left| \frac{5.9077 \times 10^{-6} - 5.9072 \times 10^{-6}}{5.9077 \times 10^{-6}} \right| \times 100 \\ &= 0.00839 \% \end{aligned}$$





Comparison Table

Order of Polynomial	1	2	3
Thermal Expansion Coefficient (in/in/°F)	5.902×10^{-6}	5.9072×10^{-6}	5.9077×10^{-6}
Absolute Relative Approximate Error	-----	0.08761%	0.00839%



Reduction in Diameter

The actual reduction in diameter is given by

$$\Delta D = D \int_{T_r}^{T_f} \alpha dT$$

where $T_r =$ room temperature ($^{\circ}F$) $T_f =$ temperature of cooling medium ($^{\circ}F$).

Since $T_r = 80^{\circ}F$ $T_f = -108^{\circ}F$

$$\Delta D = D \int_{80}^{-108} \alpha dT$$

Find out the percentage difference in the reduction in the diameter by the above integral formula and the result using the thermal expansion coefficient from the cubic interpolation.



Reduction in Diameter

We know from interpolation that

$$\alpha(T) = 6.00 \times 10^{-6} + 6.4786 * 10^{-9} T - 8.1994 * 10^{-12} T^2 + 8.1845 * 10^{-15} T^3, \quad -160 \leq T \leq 80$$

Therefore,

$$\begin{aligned} \frac{\Delta D}{D} &= \int_{T_r}^{T_f} \alpha dT \\ &= \int_{80}^{-108} (6.00 \times 10^{-6} + 6.4786 * 10^{-9} T - 8.1994 * 10^{-12} T^2 + 8.1845 * 10^{-15} T^3) dT \\ &= \left[6.00T + 6.4786 * 10^{-3} \frac{T^2}{2} - 8.1994 * 10^{-6} \frac{T^3}{3} + 8.1845 * 10^{-9} \frac{T^4}{4} \right]_{80}^{-108} \times 10^{-6} \\ &= -1105.9 \times 10^{-6} \end{aligned}$$



Reduction in diameter

Using the average value for the coefficient of thermal expansion from cubic interpolation

$$\begin{aligned}\frac{\Delta D}{D} &= \alpha \Delta T \\ &= \alpha(T_f - T_r) \\ &= 5.9077(-108 - 80) \times 10^{-6} \\ &= -1110.6 \times 10^{-6}\end{aligned}$$

The percentage difference would be

$$\begin{aligned}|\epsilon_a| &= \left| \frac{-1105.9 + 1110.6}{-1105.9} \right| \times 100 \\ &= 0.4250\%\end{aligned}$$