

Interpolation



Topic: Newton's Divided
Difference Polynomial Method

Major: Mechanical

What is Interpolation ?

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, find the value of 'y' at a value of 'x' that is not given.





Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

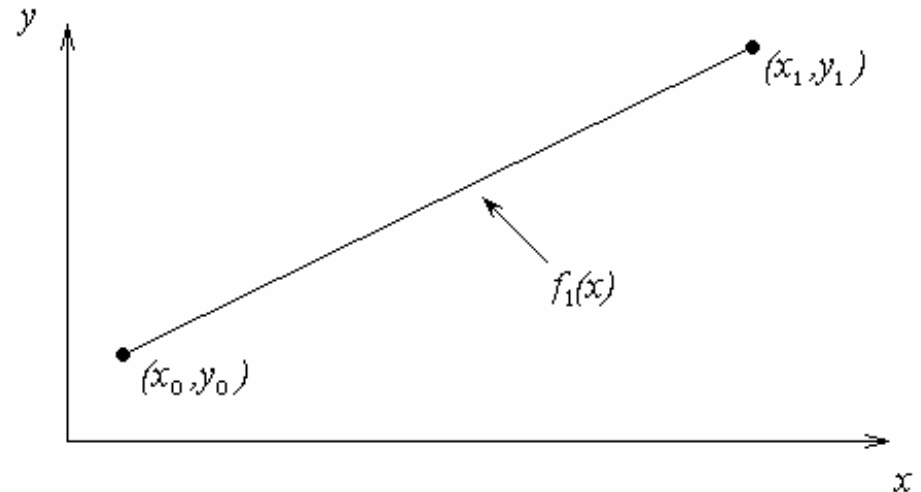
Linear interpolation: Given (x_0, y_0) , (x_1, y_1) , pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$

where

$$b_0 = f(x_0)$$

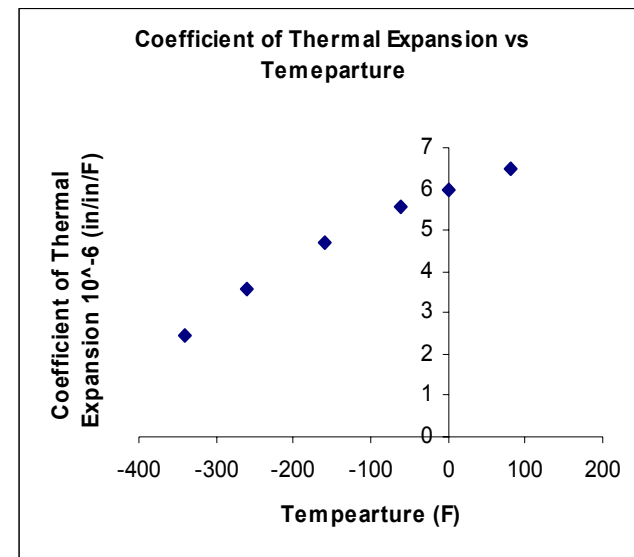
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



Example

A trunnion is cooled 80°F to -108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Find the value of the coefficient of thermal expansion at $T = -14^\circ\text{F}$ using the Newton Divided Difference method for linear interpolation.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
-60	5.58×10^{-6}
-160	4.72×10^{-6}
-260	3.58×10^{-6}
-340	2.45×10^{-6}



Linear Interpolation

$$\alpha(T) = b_0 + b_1(T - T_0)$$

$$T_0 = 0, \alpha(T_0) = 6.00 \times 10^{-6}$$

$$T_1 = -60, \alpha(T_1) = 5.58 \times 10^{-6}$$

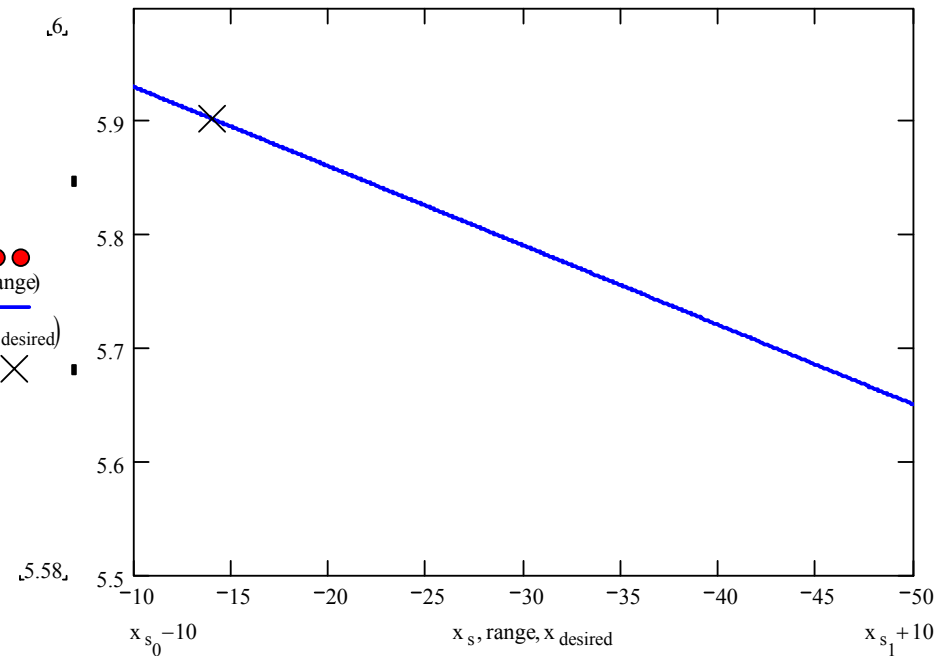
$$b_0 = \alpha(T_0)$$

$$= 6.00 \times 10^{-6}$$

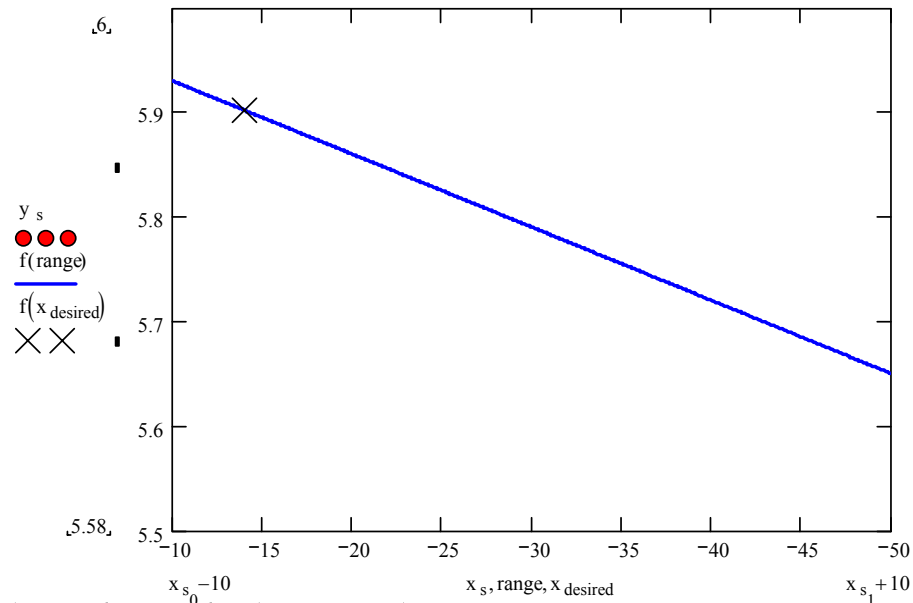
$$b_1 = \frac{\alpha(T_1) - \alpha(T_0)}{T_1 - T_0}$$

$$= \frac{5.58 \times 10^{-6} - 6.00 \times 10^{-6}}{-60 - 0}$$

$$= 0.007 \times 10^{-6}$$



Linear Interpolation (contd)



$$\begin{aligned} \alpha(T) &= b_0 + b_1(T - T_0) \\ &= 6.00 \times 10^{-6} + 0.007 \times 10^{-6}(T - 0), \\ &\quad -60 \leq T \leq 0 \end{aligned}$$

$$\alpha(-14) = 6.00 \times 10^{-6} + 0.007 \times 10^{-6}(-14 - 0)$$

$$\alpha(-14) = 5.902 \times 10^{-6} \text{ in/in/}^\circ\text{F}$$



Quadratic Interpolation

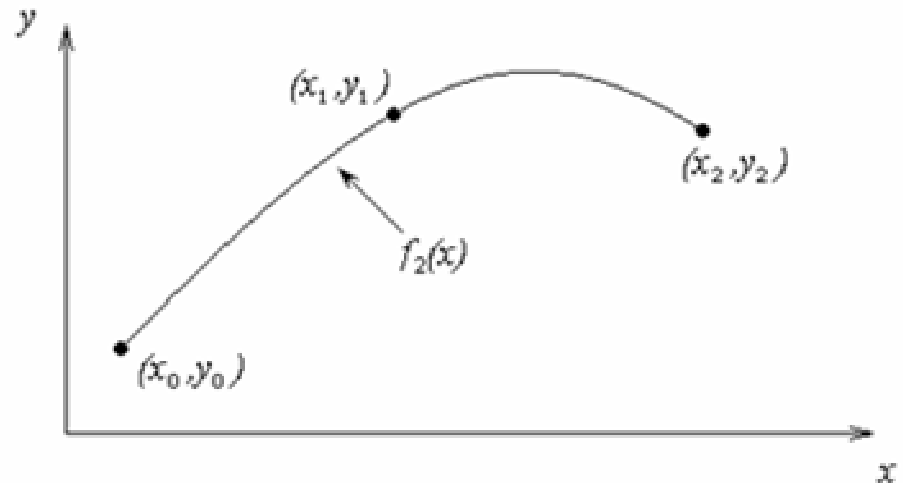
Given (x_0, y_0) , (x_1, y_1) , and (x_2, y_2) , fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

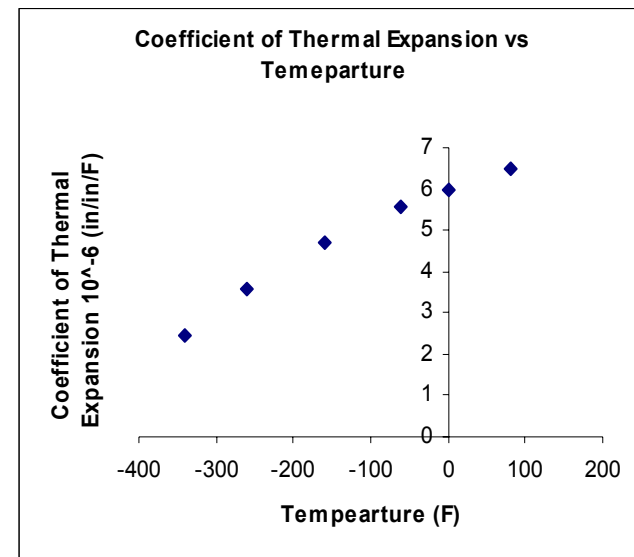
$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



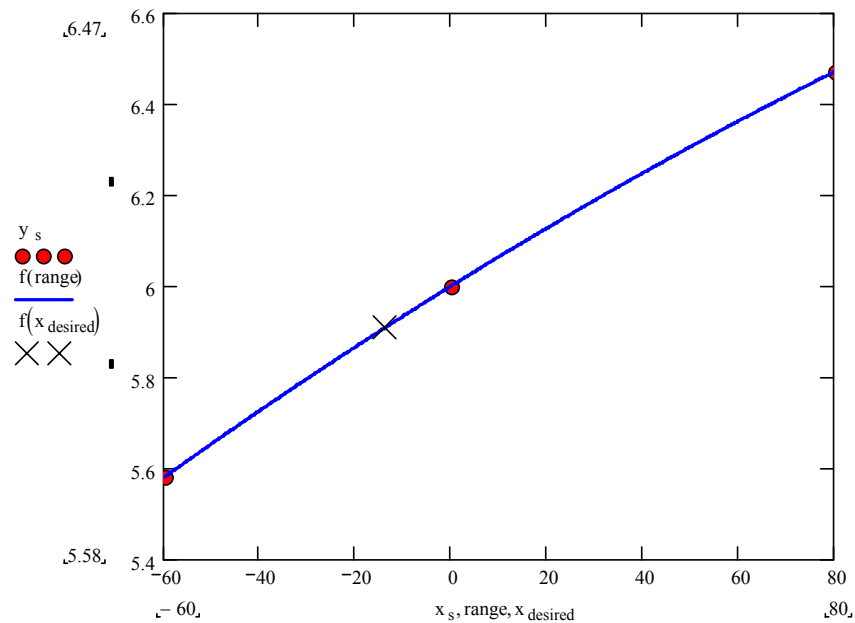
Example

A trunnion is cooled 80⁰F to -108⁰F. Given below is the table of the coefficient of thermal expansion vs. temperature. Find the value of the coefficient of thermal expansion at T=-14⁰F using the Newton Divided Difference method for quadratic polynomial.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
-60	5.58×10^{-6}
-160	4.72×10^{-6}
-260	3.58×10^{-6}
-340	2.45×10^{-6}



Quadratic Interpolation (contd)



$$\alpha(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1)$$

$$T_0 = 80, \alpha(T_0) = 6.47 \times 10^{-6}$$

$$T_1 = 0, \alpha(T_1) = 6.00 \times 10^{-6}$$

$$T_2 = -60, \alpha(T_2) = 5.58 \times 10^{-6}$$



Quadratic Interpolation (contd)

$$b_0 = \alpha(T_0)$$

$$= 6.47 \times 10^{-6}$$

$$b_1 = \frac{\alpha(T_1) - \alpha(T_0)}{T_1 - T_0} = \frac{6.00 \times 10^{-6} - 6.47 \times 10^{-6}}{0 - 80}$$

$$= 5.875 \times 10^{-9}$$

$$b_2 = \frac{\frac{\alpha(T_2) - \alpha(T_1)}{T_2 - T_1} - \frac{\alpha(T_1) - \alpha(T_0)}{T_1 - T_0}}{T_2 - T_0}$$

$$= \frac{\frac{5.58 \times 10^{-6} - 6.00 \times 10^{-6}}{-60 - 0} - \frac{6.00 \times 10^{-6} - 6.47 \times 10^{-6}}{0 - 80}}{-60 - 80}$$

$$= \frac{0.007 \times 10^{-6} - 0.005875 \times 10^{-6}}{-140} = -8.0357 \times 10^{-12}$$



Quadratic Interpolation (contd)

$$\begin{aligned}\alpha(T) &= b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) \\ &= 6.47 \times 10^{-6} + 5.875 \times 10^{-9}(T - 80) - 8.0357 \times 10^{-12}(T - 80)(T - 0), \\ &\quad -60 \leq T \leq 80\end{aligned}$$

At $T = -14$,

$$\alpha(-14) = 6.47 \times 10^{-6} + 5.875 \times 10^{-9}(-14 - 80) - 8.0357 \times 10^{-12}(-14 - 80)(-14 - 0)$$

$$\alpha(-14) = 5.9072 \times 10^{-6} \text{ in/in/}^\circ\text{F}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the first and second order polynomial is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{5.9072 \times 10^{-6} - 5.902 \times 10^{-6}}{5.9072 \times 10^{-6}} \right| \times 100 \\ &= 0.08761\%\end{aligned}$$



General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$



General Form

Given $(n + 1)$ data points, $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$,

$$f_n(x) = b_0 + b_1(x - x_0) + \dots + b_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

\vdots

$$b_{n-1} = f[x_{n-1}, x_{n-2}, \dots, x_0]$$

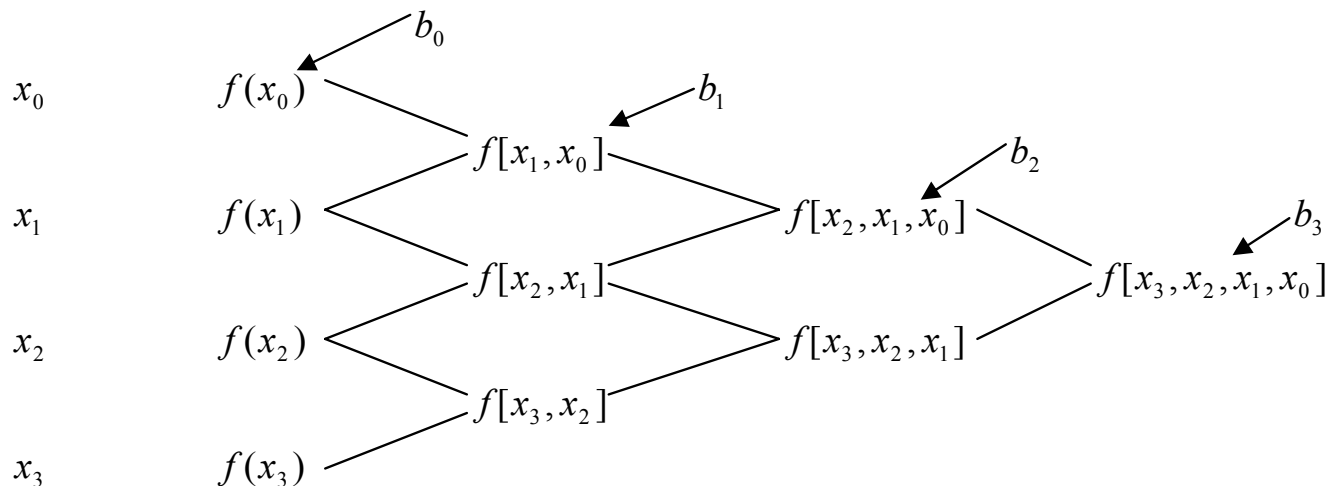
$$b_n = f[x_n, x_{n-1}, \dots, x_0]$$



General form

The third order polynomial, given (x_0, y_0) , (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) , is

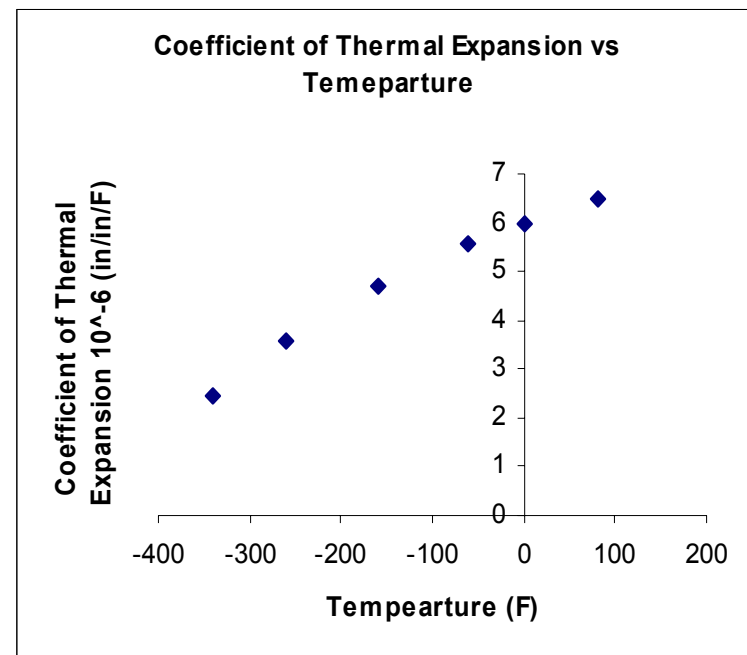
$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



Example

A trunnion is cooled 80°F to -108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Find the value of the coefficient of thermal expansion at $T = -14^\circ\text{F}$ using the Newton Divided Difference method for cubic polynomial.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
-60	5.58×10^{-6}
-160	4.72×10^{-6}
-260	3.58×10^{-6}
-340	2.45×10^{-6}





Example

The coefficient of thermal expansion profile is chosen as

$$\alpha(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) + b_3(T - T_0)(T - T_1)(T - T_2)$$

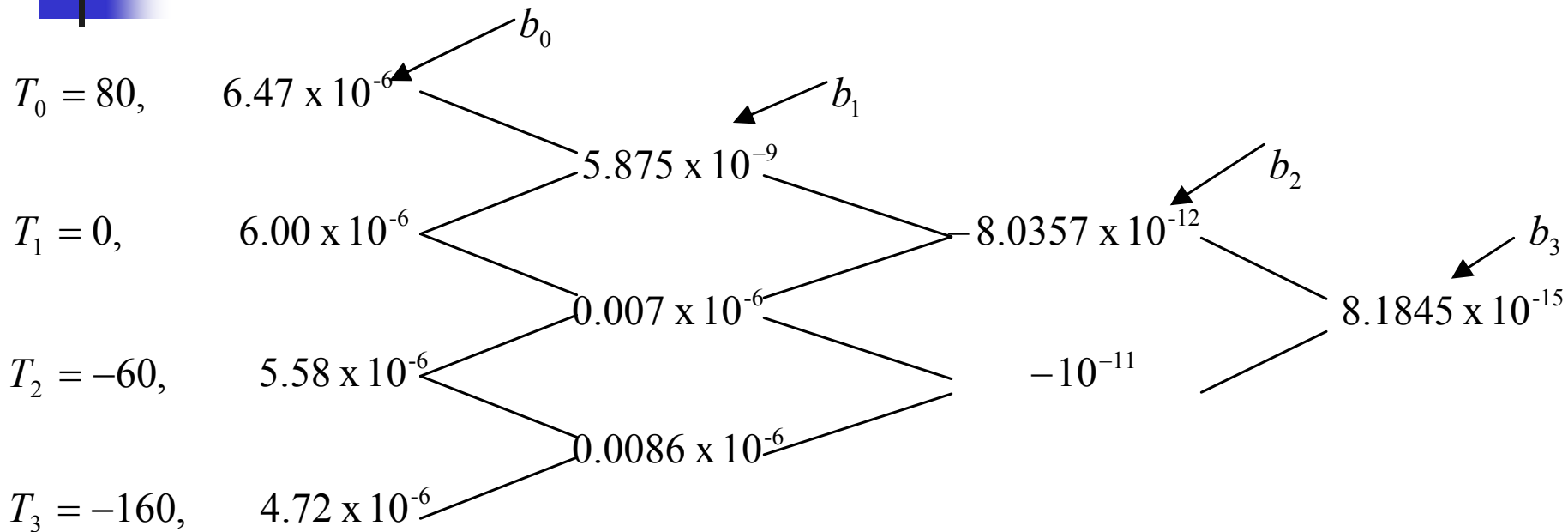
$$T_0 = 80, \quad \alpha(T_0) = 6.47 \times 10^{-6}$$

$$T_1 = 0, \quad \alpha(T_1) = 6.00 \times 10^{-6}$$

$$T_2 = -60, \quad \alpha(T_2) = 5.58 \times 10^{-6}$$

$$T_3 = -160, \quad \alpha(T_3) = 4.72 \times 10^{-6}$$

Example



The values of the constants are

$$b_0 = 6.47 \times 10^{-6} \quad b_1 = 5.875 \times 10^{-9} \quad b_2 = -8.0357 \times 10^{-12} \quad b_3 = 8.1845 \times 10^{-15}$$



Example

$$\begin{aligned}\alpha(T) &= b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) + b_3(T - T_0)(T - T_1)(T - T_2) \\ &= 6.47 \times 10^{-6} + 5.875 \times 10^{-9}(T - 80) - 8.0357 \times 10^{-12}(T - 80)(T - 0) \\ &\quad + 8.8145 \times 10^{-15}(T - 80)(T - 0)(T + 60)\end{aligned}$$

At $T = -14$,

$$\begin{aligned}\alpha(-14) &= 6.47 \times 10^{-6} + 5.875 \times 10^{-9}(-14 - 80) - 8.0357 \times 10^{-12}(-14 - 80)(-14 - 0) \\ &\quad + 8.1845 \times 10^{-15}(-14 - 80)(-14 - 0)(-14 + 60) \\ \alpha(-14) &= 5.9077 * 10^{-6} \text{ in / in / } ^\circ F\end{aligned}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the second and third order polynomial is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{5.9077 \times 10^{-6} - 5.9072 \times 10^{-6}}{5.9077 \times 10^{-6}} \right| \times 100 \\ &= 0.00839\%\end{aligned}$$



Comparison Table

Order of Polynomial	1	2	3
Thermal Expansion Coefficient (in/in/°F)	5.902×10^{-6}	5.9072×10^{-6}	5.9077×10^{-6}
Absolute Relative Approximate Error	-----	0.08761%	0.00839%



Reduction in Diameter

The actual reduction in diameter is given by

$$\Delta D = D \int_{T_r}^{T_f} \alpha dT$$

where $T_r =$ room temperature ($^{\circ}F$) $T_f =$ temperature of cooling medium ($^{\circ}F$).

Since $T_r = 80^{\circ}F$ $T_f = -108^{\circ}F$

$$\Delta D = D \int_{80}^{-108} \alpha dT$$

Find out the percentage difference in the reduction in the diameter by the above integral formula and the result using the thermal expansion coefficient from the cubic interpolation.



Reduction in Diameter

We know from Cubic Interpolation that

$$\alpha(T) = 6.00 \times 10^{-6} + 6.4786 \times 10^{-9} T - 8.1994 \times 10^{-12} T^2 + 8.1845 \times 10^{-15} T^3, \quad -160 \leq T \leq 80,$$

Therefore,

$$\begin{aligned} \frac{\Delta D}{D} &= \int_{T_r}^{T_f} \alpha dT \\ &= \int_{80}^{-108} (6.00 \times 10^{-6} + 6.4786 \times 10^{-9} T - 8.1994 \times 10^{-12} T^2 + 8.1845 \times 10^{-15} T^3) dT \\ &= \left[6.00 \times 10^{-6} T + 6.4786 \times 10^{-9} \frac{T^2}{2} - 8.1994 \times 10^{-12} \frac{T^3}{3} + 8.1845 \times 10^{-15} \frac{T^4}{4} \right]_{80}^{-108} \end{aligned}$$

$$\frac{\Delta D}{D} = -1105.9 * 10^{-6} \text{ in / in}$$



Reduction in diameter

Using the average value for the coefficient of thermal expansion from Cubic Interpolation

$$\begin{aligned}\frac{\Delta D}{D} &= \alpha \Delta T \\ &= \alpha(T_f - T_r) \\ &= 5.9077 \times 10^{-6}(-108 - 80)\end{aligned}$$

$$\frac{\Delta D}{D} = -1110.6 * 10^{-6} \text{ in / in}$$

Considering the integral to be more accurate, the percentage difference would be

$$\begin{aligned}|\epsilon_a| &= \left| \frac{-1105.9 \times 10^{-6} + 1110.6 \times 10^{-6}}{-1105.9 \times 10^{-6}} \right| \times 100 \\ &= 0.4250\%\end{aligned}$$