

# Interpolation



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## Topic: Spline Interpolation Method

Major: Mechanical



# What is Interpolation ?

Given  $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ , find the value of 'y' at a value of 'x' that is not given.





# Interpolants

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Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

# Why Splines ?

$$f(x) = \frac{1}{1 + 25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

$x$	$y = \frac{1}{1 + 25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

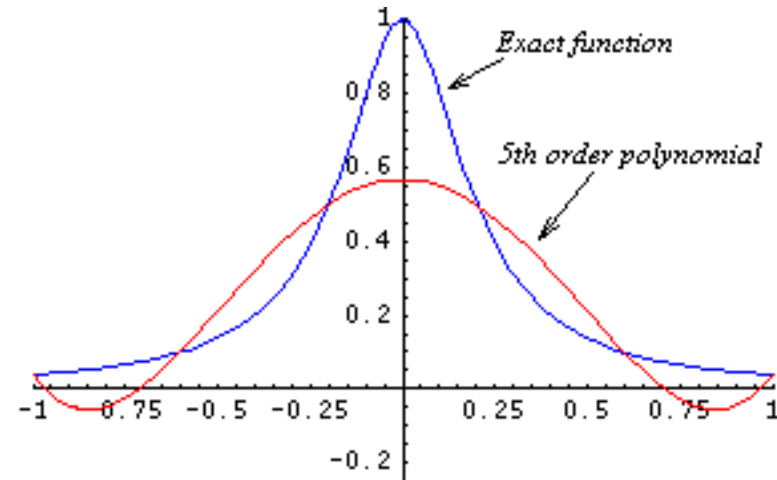
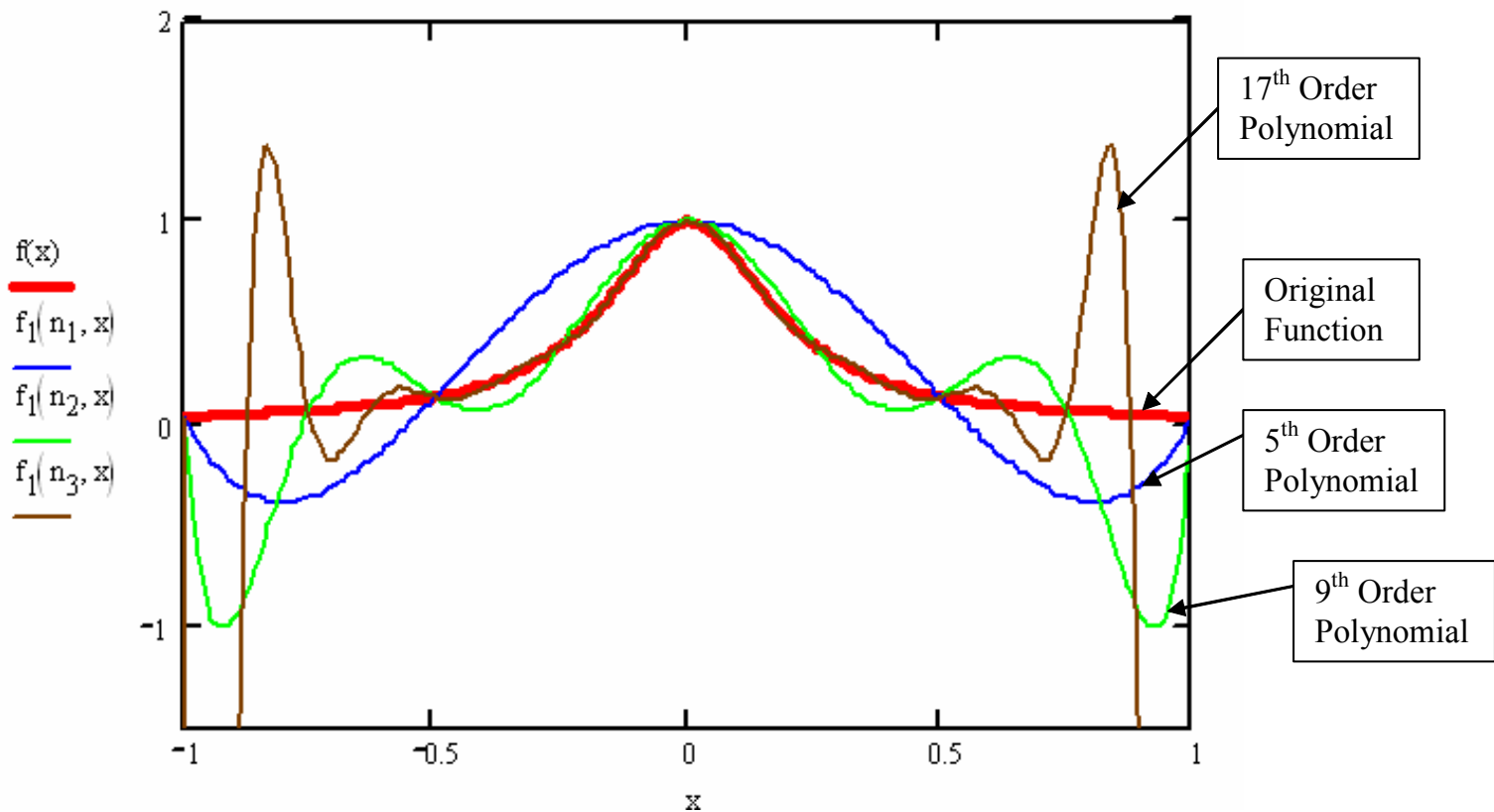


Figure : 5<sup>th</sup> order polynomial vs. exact function

# Why Splines ?



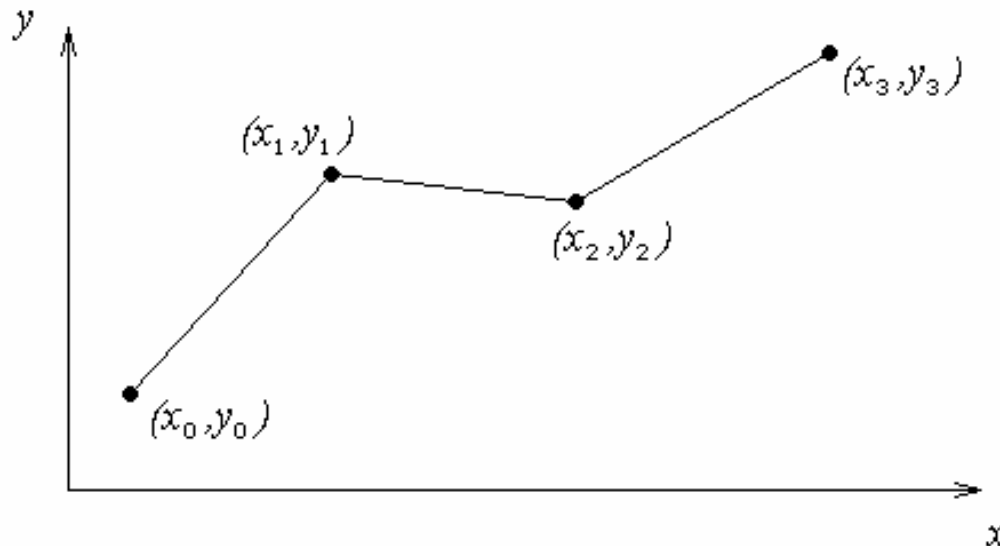
**Figure : Higher order polynomial interpolation is a bad idea**



# Linear Interpolation

Given  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by  $(y_i = f(x_i))$

**Figure : Linear splines**





# Linear Interpolation (contd)

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$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \quad x_0 \leq x \leq x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \quad x_1 \leq x \leq x_2$$

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$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \leq x \leq x_n$$

Note the terms of

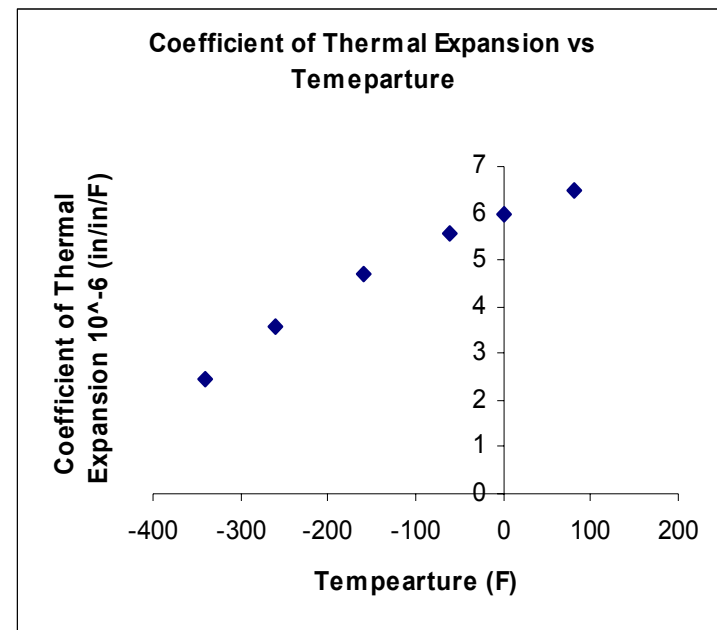
$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between  $x_{i-1}$  and  $x_i$ .

# Example

A trunnion is cooled 80°F to -108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Find the value of the coefficient of thermal expansion at  $T = -14^\circ\text{F}$  using the linear spline Interpolation method.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	$6.47 \times 10^{-6}$
0	$6.00 \times 10^{-6}$
-60	$5.58 \times 10^{-6}$
-160	$4.72 \times 10^{-6}$
-260	$3.58 \times 10^{-6}$
-340	$2.45 \times 10^{-6}$



# Linear Interpolation

$$T_0 = 0, \quad \alpha(T_0) = 6.00 \times 10^{-6}$$

$$T_1 = -60, \quad \alpha(T_1) = 5.58 \times 10^{-6}$$

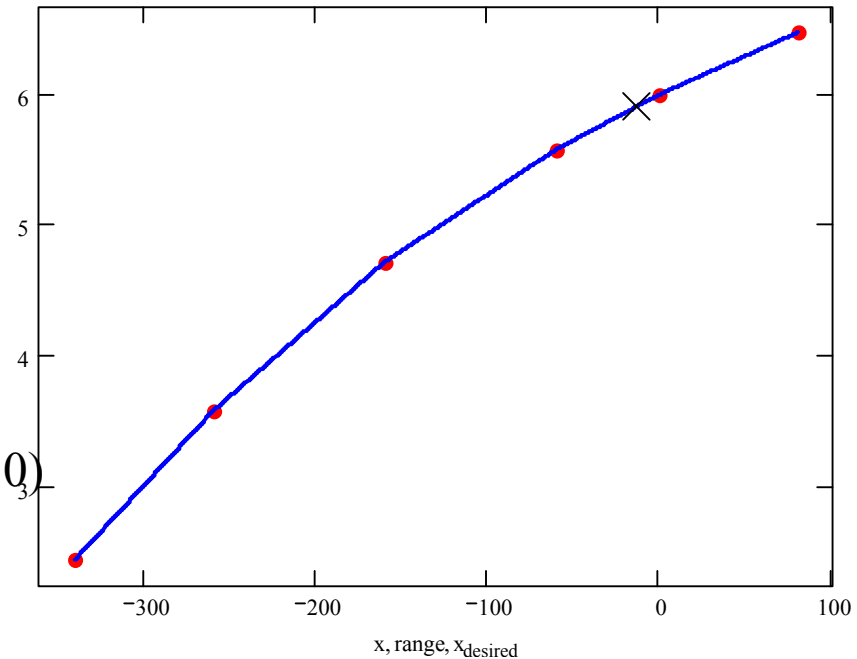
$$\alpha(T) = \alpha(T_0) + \frac{\alpha(T_1) - \alpha(T_0)}{T_1 - T_0} (T - T_0)$$
$$= 6.00 \times 10^{-6} + \frac{5.58 \times 10^{-6} - 6.00 \times 10^{-6}}{-60 - 0} (T - 0)$$

$$\alpha(T) = 6.00 \times 10^{-6} + 0.007 \times 10^{-6} (T - 0)$$

At  $T = -14$ ,

$$\alpha(-14) = 6.00 \times 10^{-6} + 0.007 \times 10^{-6} (-14 - 0)$$

$$\alpha(-14) = 5.902 \times 10^{-6} \text{ in/in/}^\circ\text{F}$$



# Quadratic Interpolation

Given  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , fit quadratic splines through the data. The splines are given by

$$f(x) = a_1x^2 + b_1x + c_1, \quad x_0 \leq x \leq x_1$$

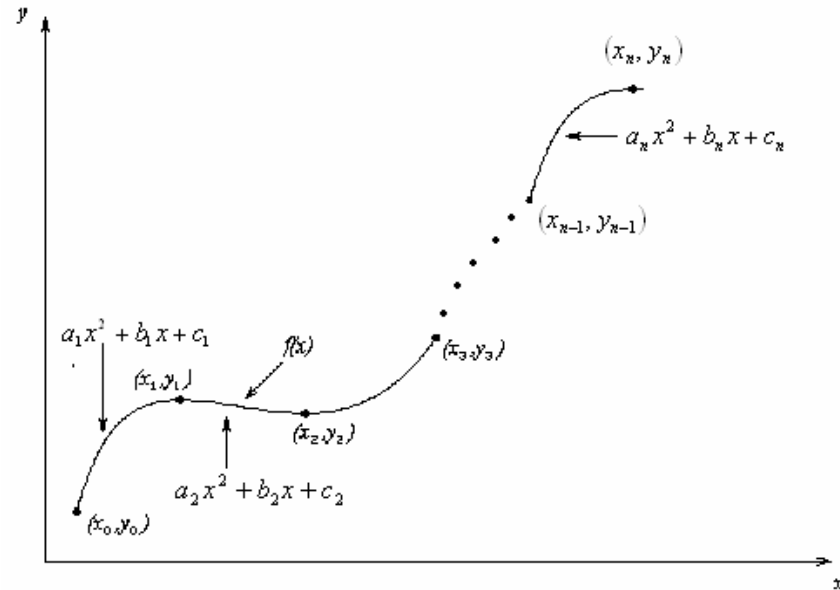
$$= a_2x^2 + b_2x + c_2, \quad x_1 \leq x \leq x_2$$

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$$= a_nx^2 + b_nx + c_n, \quad x_{n-1} \leq x \leq x_n$$



Find  $a_i, b_i, c_i, i = 1, 2, \dots, n$

# Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$

⋮

⋮

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

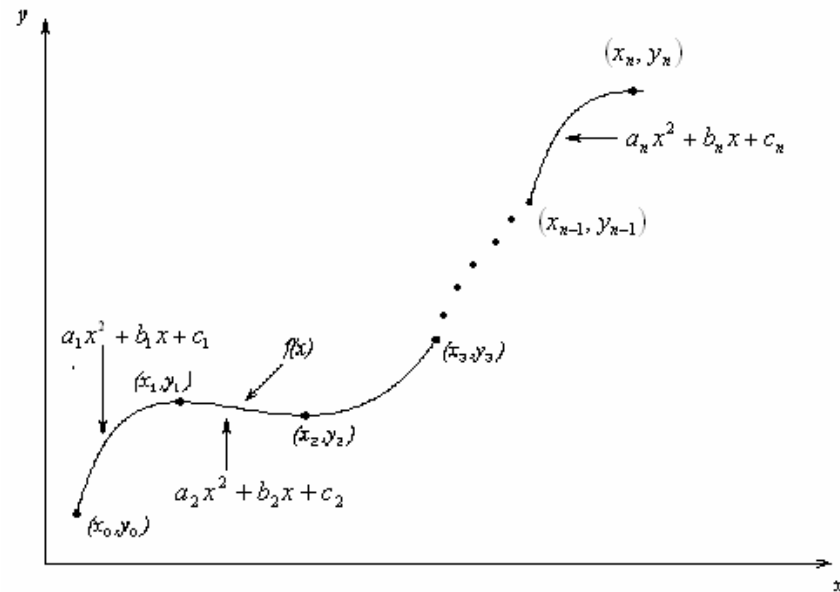
$$a_i x_i^2 + b_i x_i + c_i = f(x_i)$$

⋮

⋮

$$a_n x_{n-1}^2 + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$



This condition gives  $2n$  equations

# Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \text{ is } 2a_1x + b_1$$

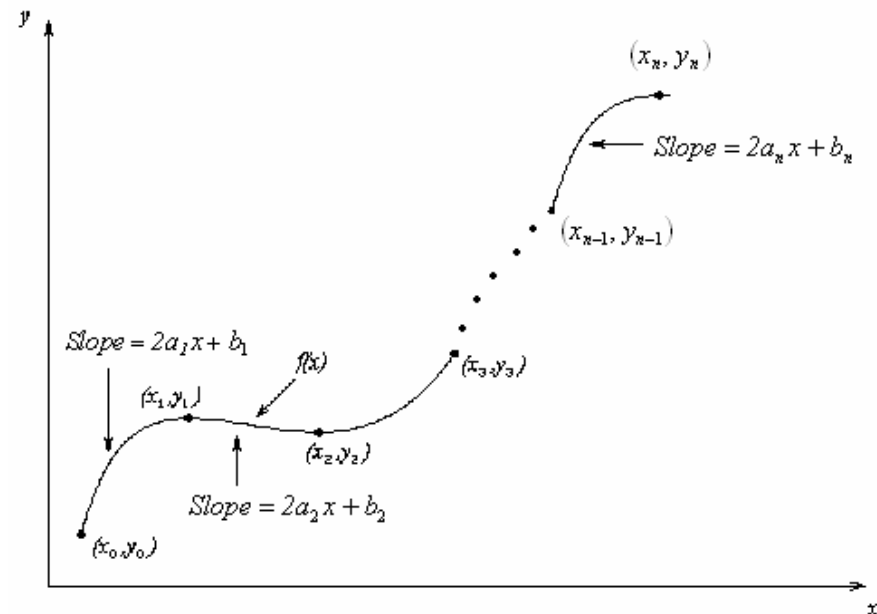
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \text{ is } 2a_2x + b_2$$

and the two are equal at  $x = x_1$  giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



# Quadratic Splines (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

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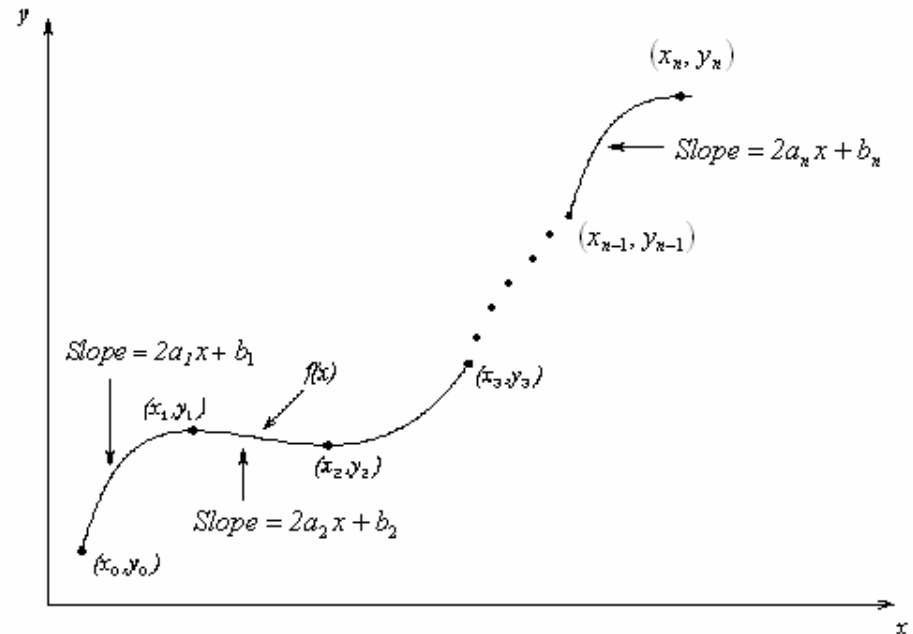
$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

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$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



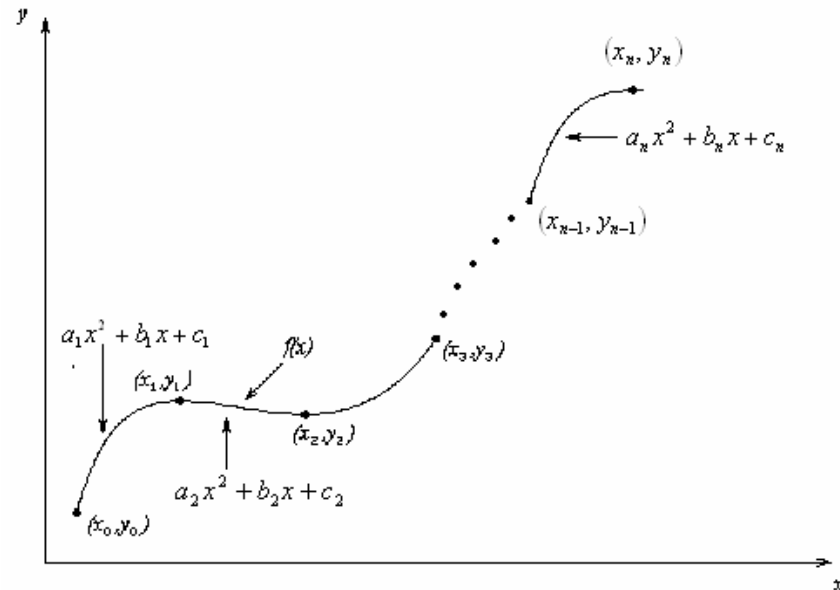
We have (n-1) such equations. The total number of equations is  $(2n) + (n - 1) = (3n - 1)$ .

We can assume that the first spline is linear, that is  $a_1 = 0$

# Quadratic Splines (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

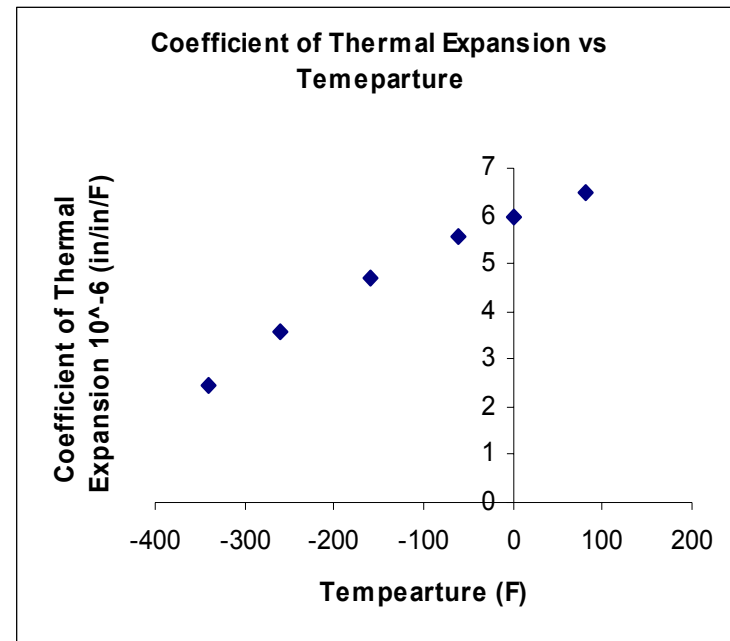
$$\begin{aligned}
 f(x) &= a_1x^2 + b_1x + c_1, & x_0 \leq x \leq x_1 \\
 &= a_2x^2 + b_2x + c_2, & x_1 \leq x \leq x_2 \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 &= a_nx^2 + b_nx + c_n, & x_{n-1} \leq x \leq x_n
 \end{aligned}$$



# Example

A trunnion is cooled 80°F to -108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Find the value of the coefficient of thermal expansion at  $T = -14^\circ\text{F}$  using the quadratic spline interpolation method.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	$6.47 \times 10^{-6}$
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-60	$5.58 \times 10^{-6}$
-160	$4.72 \times 10^{-6}$
-260	$3.58 \times 10^{-6}$
-340	$2.45 \times 10^{-6}$



# Solution

Since there are six data points,

five quadratic splines pass through them.

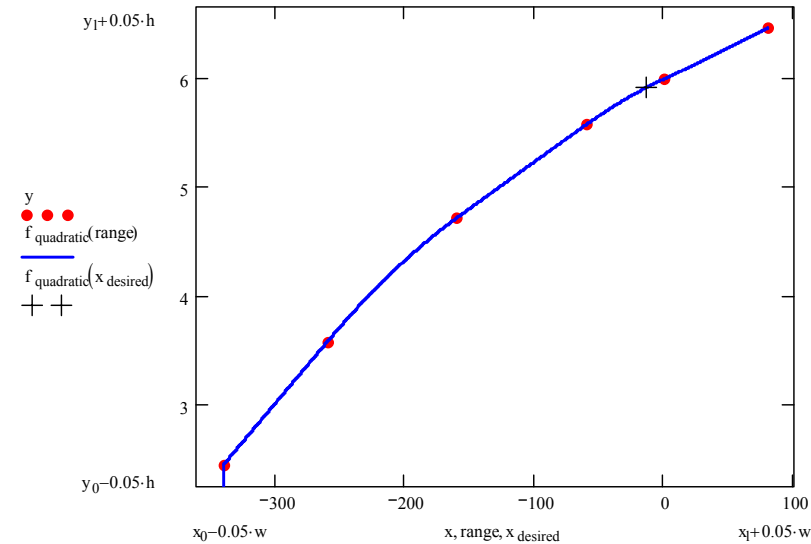
$$\alpha(T) = a_1 T^2 + b_1 T + c_1, \quad -340 \leq T \leq -260$$

$$= a_2 T^2 + b_2 T + c_2, \quad -260 \leq T \leq -160$$

$$= a_3 T^2 + b_3 T + c_3, \quad -160 \leq T \leq -60$$

$$= a_4 T^2 + b_4 T + c_4, \quad -60 \leq T \leq 0$$

$$= a_5 T^2 + b_5 T + c_5, \quad 0 \leq T \leq 80$$





# Solution (contd)

Setting up the equations

Each quadratic spline passes through two consecutive data points giving

$a_1T^2 + b_1T + c_1$  passes through  $T = -340$  and  $T = -260$ ,

$$a_1(-340)^2 + b_1(-340) + c_1 = 2.45 \times 10^{-6} \quad (1)$$

$$a_1(-260)^2 + b_1(-260) + c_1 = 3.58 \times 10^{-6} \quad (2)$$

Similarly,

$$a_2(-260)^2 + b_2(-260) + c_2 = 3.58 \times 10^{-6} \quad (3)$$

$$a_2(-160)^2 + b_2(-160) + c_2 = 4.72 \times 10^{-6} \quad (4)$$

$$a_3(-160)^2 + b_3(-160) + c_3 = 4.72 \times 10^{-6} \quad (5)$$

$$a_3(-60)^2 + b_3(-60) + c_3 = 5.58 \times 10^{-6} \quad (6)$$

$$a_4(-60)^2 + b_4(-60) + c_4 = 5.58 \times 10^{-6} \quad (7)$$

$$a_4(0)^2 + b_4(0) + c_4 = 6.00 \times 10^{-6} \quad (8)$$

$$a_5(0)^2 + b_5(0) + c_5 = 6.00 \times 10^{-6} \quad (9)$$

$$a_5(80)^2 + b_5(80) + c_5 = 6.47 \times 10^{-6} \quad (10)$$



# Solution (contd)

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Quadratic splines have continuous derivatives at the interior data points

At  $T = -260$

$$2a_1(-260) + b_1 - 2a_2(-260) - b_2 = 0 \quad (11)$$

At  $T = -160$

$$2a_2(-160) + b_2 - 2a_3(-160) - b_3 = 0 \quad (12)$$

At  $T = -60$

$$2a_3(-60) + b_3 - 2a_4(-60) - b_4 = 0 \quad (13)$$

At  $T = 0$

$$2a_4(0) + b_4 - 2a_5(0) - b_5 = 0 \quad (14)$$

Assuming the first spline  $a_1T^2 + b_1T + c_1$  is linear,

$$a_1 = 0 \quad (15)$$

# Solution (contd)

$$\begin{bmatrix}
 1.156 \times 10^5 & -340 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 67600 & -260 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 67600 & -260 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 25600 & -160 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 25600 & -160 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 3600 & -60 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3600 & -60 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6400 & 80 & 1 & 0 \\
 -520 & 1 & 0 & 520 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -320 & 1 & 0 & 320 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & -120 & 1 & 0 & 120 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 a_1 \\
 b_1 \\
 c_1 \\
 a_2 \\
 b_2 \\
 c_2 \\
 a_3 \\
 b_3 \\
 c_3 \\
 a_4 \\
 b_4 \\
 c_4 \\
 a_5 \\
 b_5 \\
 c_5
 \end{bmatrix}
 =
 \begin{bmatrix}
 2.45 \times 10^{-6} \\
 3.58 \times 10^{-6} \\
 3.58 \times 10^{-6} \\
 4.72 \times 10^{-6} \\
 4.72 \times 10^{-6} \\
 5.58 \times 10^{-6} \\
 5.58 \times 10^{-6} \\
 6.00 \times 10^{-6} \\
 6.00 \times 10^{-6} \\
 6.47 \times 10^{-6} \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}$$



# Solution (contd)

Solving the above 15 equations gives the 15 unknowns as

$i$	$a_i$	$b_i$	$c_i$
1	0	$0.014125 \times 10^{-6}$	$7.2525 \times 10^{-6}$
2	$-2.725 \times 10^{-11}$	$-4.5 \times 10^{-11}$	$5.4104 \times 10^{-6}$
3	$-7.5 \times 10^{-13}$	$0.008435 \times 10^{-6}$	$6.0888 \times 10^{-6}$
4	$-2.5417 \times 10^{-11}$	$0.005475 \times 10^{-6}$	$6 \times 10^{-6}$
5	$5 \times 10^{-12}$	$0.005475 \times 10^{-6}$	$6 \times 10^{-6}$



# Solution (contd)

Therefore, the splines are given by

$$\begin{aligned}\alpha(T) &= 0.014125 \times 10^{-6} T + 7.2525 \times 10^{-6}, & -340 \leq T \leq -260 \\ &= -2.725 \times 10^{-11} T^2 - 4.5 \times 10^{-11} T + 5.4104 \times 10^{-6}, & -260 \leq T \leq -160 \\ &= -7.5 \times 10^{-13} T^2 + 0.008435 \times 10^{-6} T + 6.0888 \times 10^{-6}, & -160 \leq T \leq -60 \\ &= -2.5417 \times 10^{-11} T^2 + 0.005475 \times 10^{-6} T + 6 \times 10^{-6}, & -60 \leq T \leq 0 \\ &= 5 \times 10^{-12} T^2 + 0.005475 \times 10^{-6} T + 6 \times 10^{-6}, & 0 \leq T \leq 80\end{aligned}$$

At  $T = -14$

$$\alpha(-14) = -2.5417 \times 10^{-11} (-14)^2 + 0.005475 \times 10^{-6} (-14) + 6 \times 10^{-6}$$

$$\alpha(-14) = 5.9184 \times 10^{-6} \text{ in/in}/^\circ F$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the first order and second order polynomial is

$$|\epsilon_a| = \left| \frac{5.8761 \times 10^{-6} - 5.9184 \times 10^{-6}}{5.8761 \times 10^{-6}} \right| \times 100$$

$$= 0.71987\%$$



# Reduction in Diameter

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The actual reduction in diameter is given by

$$\Delta D = D \int_{T_r}^{T_f} \alpha dT$$

where  $T_r =$  room temperature ( $^{\circ}F$ )  $T_f =$  temperature of cooling medium ( $^{\circ}F$ ).

Since  $T_r = 80^{\circ}F$   $T_f = -108^{\circ}F$

$$\Delta D = D \int_{80}^{-108} \alpha dT$$

Find out the percentage difference in the reduction in the diameter by the above integral formula and the result using the thermal expansion coefficient from the cubic interpolation.

# Reduction in Diameter

$$\begin{aligned}
 \int_{T_r}^{T_f} \alpha dT &= \int_{80}^{-108} \alpha(T) dT = \int_{-60}^{-108} \alpha(T) dT + \int_0^{-60} \alpha(T) dT + \int_{80}^0 \alpha(T) dT \\
 &= \int_{-60}^{-108} (-7.5 \times 10^{-13} T^2 + 0.008435 \times 10^{-6} T + 6.0888 \times 10^{-6}) dT \\
 &\quad + \int_0^{-60} (-2.5417 \times 10^{-11} T^2 + 0.005475 \times 10^{-6} T + 6 \times 10^{-6}) dT \\
 &\quad + \int_{80}^0 (5 \times 10^{-12} T^2 + 0.005475 \times 10^{-6} T + 6 \times 10^{-6}) dT \\
 &= \left[ -7.5 \times 10^{-13} \frac{T^3}{3} + 0.008435 \times 10^{-6} \frac{T^2}{2} + 6.0888 \times 10^{-6} T \right]_{-60}^{-108} \\
 &\quad + \left[ -2.5417 \times 10^{-11} \frac{T^3}{3} + 0.005475 \times 10^{-6} \frac{T^2}{2} + 6 \times 10^{-6} T \right]_0^{-60} \\
 &\quad + \left[ 5 \times 10^{-12} \frac{T^3}{3} + 0.005475 \times 10^{-6} \frac{T^2}{2} + 6 \times 10^{-6} T \right]_{80}^0 \\
 &= [-257.992] \times 10^{-6} + [-348.315] \times 10^{-6} + [-498.373] \times 10^{-6}
 \end{aligned}$$

$$\int_{T_r}^{T_f} \alpha dT = -1104.7 \times 10^{-6} \text{ in/in}$$



# Reduction in diameter

Taking the average coefficient of thermal expansion over this interval, given by:

$$\begin{aligned}\alpha_{ave} &= \frac{\int_{T_r}^{T_f} \alpha dT}{T_f - T_r} \\ &= \frac{-1104.7 \times 10^{-6}}{-108 - 80} \\ &= 5.8761 \times 10^{-6} \\ \alpha_{ave} &= 5.8761 * 10^{-6} \text{ in / in / } ^\circ F\end{aligned}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the

2 methods is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{5.8761 \times 10^{-6} - 5.9184 \times 10^{-6}}{5.8761 \times 10^{-6}} \right| \times 100 \\ &= 0.71987\%\end{aligned}$$