Differentiation of Discrete Functions

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Introduction

This worksheet demonstrates the use of Maple to illustrate the differentiation of discrete functions using:

a) Forward Divided Difference Method. To find the derivative of a function given at discrete n +1 data points (x_1,y_1), (x_2,y_2),...,(x_n ,y_n), the value of the f'(x) for $x_i \le x \le x_{i+1}$, i=1,...,n-1, is given by

$$f'(x_i) \cong \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

b) By differentiation of a second order interpolated polynomial

c) By differentiation of a polynomial using all data points

Section 1: Input

The following simulation approximates the first derivative of a discrete function using Forward Divided Difference. The user inputs are

a) Data with *x* and *y* values. At least 3 data points are required.

b) point at which the derivative is to be found, *xv*

The outputs include

a) approximate value of the derivative at the given point using Forward Divided Difference

b) approximate value of the derivative at the given point by differentiating a second order polynomial found using the three closest values to *xv*, and that bracket *xv*.

c) approximate value of the derivative at the given point using all data points and differentiating a $n\text{-}1^{th}$ order polynomial that goes through the *n* data points

Data points, *y* vs. *x*

```
x := {10, 15, 20, 22, 25};
y := {100, 225, 400, 484, 625};
```

Value of x at which *f '(x)* is desired, *xv*

```
xv = 18;
```

This is the end of the user section. All the information must be entered before proceeding to the next section.

Section 2: Calculation

The following loop estimates the solution of first derivate of a discrete function at a point *xv* using Forward Divided Difference method. The loop above checks if the value at which the solution is desired is between *x[1]* and *x[n]*. If the value is between *x[1]* and *x[n]*, then the slope from the closest points that bracket the value is found. The value of the slope is the value of the derivative at that point.
n = number of data points

```
n := Length[x];
If[x[[1]] ≤ xv ≤ x[[n]], Do[
  If[x[[i]] ≤ xv ≤ x[[i + 1]],
   AV = (y[[i + 1]] - y[[i]]) / (x[[i + 1]] - x[[i]]);
  ]
  , {i, 1, n - 1, 1}], Print["Point where derivative was requested is outside the domain of x"]
]
```

The next method takes the three closest points to the given value to find a second order polynomial and differentiates it to find the value of *f'(x)* at *x=xv*.

```
Needs["Splines`"]
If[x[[1]] ≤ xv ≤ x[[n]],
 If [x[[1]] ≤ xv < x[[2]],
   data = Table[{x[[i]], y[[i]]}, {i, 1, 3}];
   Sol = D[ Fit[data, {1, a, a^2, a^3}, a], a] /. a → xv;
  ]
  If [x[[n - 1]] ≤ xv < x[[n]],
   data = Table[{x[[i]], y[[i]]}, {i, n - 2, n}];
   Sol = D[ Fit[data, {1, a, a^2, a^3}, a], a] /. a → xv;
  ], Do[
  If[x[[b]] ≤ xv < x[[b + 1]],
   If[Abs[x[[b + 2]] - xv] ≤ Abs[xv - x[[b - 1]]],
    data = Table[{x[[i]], y[[i]]}, {i, b, b + 2}];
    Sol = D[ Fit[data, {1, a, a^2, a^3}, a], a] /. a → xv;
    , data = Table[{x[[i]], y[[i]]}, {i, b - 1, b + 1}];
    Sol = D[ Fit[data, {1, a, a^2, a^3}, a], a] /. a → xv;
   ]]
  , {b, 2, n - 2, 1}]
]
data
Sol
```

Null^2

```
{{15, 225}, {20, 400}, {22, 484}}
```

```
35.9164
```

Next, all data points are going to be used to find a n-1 order polynomial. At the end, the polynomial is going to be differentiated and the value at which the derivative is wished to be found is going to be found.

```
data = Table[{x[[i]], y[[i]]}, {i, 1, n}];
Solv = D[InterpolatingPolynomial[data, c], c] /. c → xv;
```

Section 3: Spreadsheet

The next table shows the approximate values of the derivative at the given point using Forward Divided Difference, second and n-1th order polynomials.

```
Print["     ", "Straight Line", "        ", "Second Order", "        ", "n-1th Order"]
Print[" "]
Print["       ", AV, "                  ", Sol, "               ", Solv]
```

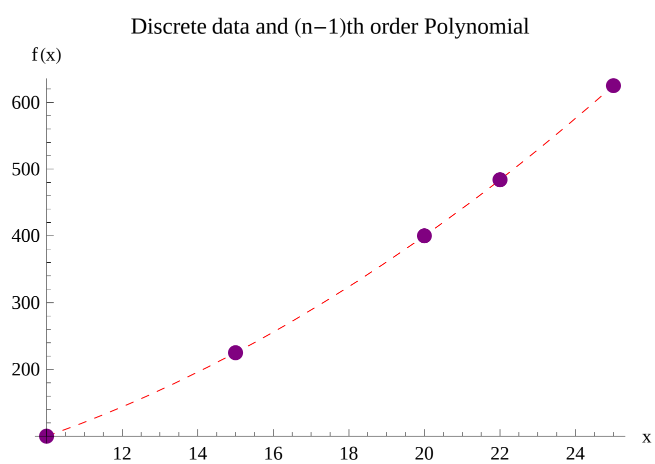
```
                    Straight Line        Second Order        n-1th Order


                         35                 35.9164              36
```

Section 4: Graphs

The following graph shows the discrete data, linear splines and the n-1th order polynomial.

```
d = Fit[data, {1, a, a^2, a^3}, a];
Show[Plot[d, {a, x[[1]], x[[n]]}, PlotStyle → {Red, Dashed}],
 ListPlot[data, PlotStyle → Directive[PointSize[Large], Purple]],
 PlotLabel → "Discrete data and (n-1)th order Polynomial", AxesLabel → {"x", "f(x)"}]
```



References

Numerical Differentiation of Discrete Functions.
See http://numericalmethods.eng.usf.edu/mws/gen/02dif

Questions

1. The thermal expansion coefficient of steel is a function of temperature. Find the rate of change of the thermal expansion coefficient as a function of temperature at T=-200 F. Is this rate of change at T=-200 F more or less than that at T=50 F? Use Forward Divided Difference to answer.

2. The distance traveled by a rocket is given as a function of time

t, s	0	10	20	30	40
x, miles	0	16	28	39	53

Find the rocket velocity and acceleration at t=25s using numerical differentiation.

Conclusions

The more data points taken to obtain the first derivative of a discrete function, more accurate the approximate value is. However, the more data points taken implies taking the risk of also having an innacurate value.